Non-standard Backward Stochastic Differential Equations and Multiple Optimal Stopping Problems with Applications to Securities Pricing

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Abstract

This thesis elaborates on several topics on the wealth maximization problem of a small investor who invests in a financial market. Key tools for our studies come across in the form of several classes of BSDEs with particular non-linearities, casting them outside the standard class of Lipschitz continuous BSDEs. These non-standard BSDEs appear to arise naturally in the domain of stochastic control problems. We first give a characterization of a small investor's optimal wealth and its associated optimal strategy by means of a systems of coupled equations, a forward-backward stochastic differential equation (FBSDE) with non-Lipschitz coefficients, where the backward component is of quadratic growth. This characterization is based on verification arguments and makes use of other approaches to this problem, including the stochastic maximum principle and the convex duality approach. We proceed by establishing a solution concept for this induced FBSDE by means of a compactness result for bounded martingales.

We then examine how specifying concrete utility functions give rise to another class of non-standard BSDEs. In this context, we also investigate the relationship to a modeling approach based on random fields techniques, known by now as the backward stochastic partial differential equations (BSPDEs) approach. It turns out that by specifying the investor's utility to be of exponential, logarithmic and power type, the feature of separate time and space components of the random field appears. One of the consequence of this separability property is that solving the BSPDE effectively boils down to solving an ordinary (yet still non-standard) BSDE. In this context, we present several financial applications and also discuss their numerical treatment.

We continue with the presentation of a numerical method for a special type of quadratic BSDEs. This method is based on a stochastic analogue to the Cole-Hopf transformation from PDE theory. We discuss its applicability to numerically solve indifference pricing problems for contingent claims in an incomplete market.

We then proceed to BSDEs whose drifts explicitly incorporate path dependence. Several analytical properties for this type of non-standard BSDEs are derived. In particular, we obtain a path regularity result for such type of BSDEs, a property which has the potential to be exploited for the design of an implementable numerical algorithm.

In the last part, we devote our attention to the problem of a small investor who is equipped with several exercise rights that allow her to collect pre-specified cash-flows. This type of multi-exercise options is encountered e.g. when one considers swing options on electricity markets. The investor is seeking to exercise her rights in a way which yields the best possible outcome for her. However, she has to abide to the rules that a certain set of constraints on her exercise behavior are imposed in the contract. On the other side, the seller of such a product is interested in charging the amount of money which arises as the discounted value of the best possible outcome from optimally exercising. Hence, we face the problem of giving such multi-exercise contracts a fair price. We solve this problem by casting it into the language of multiple optimal stopping and develop a martingale dual approach for characterizing the optimal possible outcome. Moreover, we develop regression based Monte Carlo algorithms which simulate efficiently lower and upper price bounds. Finally, we present a numerical study in which we give tight confidence intervals for the price of swing options incorporating refraction periods and volume constraints.

Zusammenfassung

Zentraler Gegenstand dieser Dissertation ist die Entwicklung von mathematischen Methoden zur Charakterisierung und Implementierung von optimalen Investmentstrategien eines Kleininvestors auf einem Finanzmarkt. Zur Behandlung dieser Probleme ziehen wir als Hauptwerkzeug Stochastische Rückwärts-Differenzialgleichungen (BSDEs) mit nicht-linearen Drifts heran. Diese Nicht-Lineariäten ordnen sie außerhalb der Standardklasse der Lipschitz-stetigen BSDEs ein und treten häufig in finanzmathematischen Kontrollproblemen auf.
Zunächst charakterisieren wir das optimale Vermögen und die optimale Investmentstrategie eines Kleininvestors mit Hilfe einer sog. Stochastischen Vorwärts-Rückwärts-Differenzialgleichung (FBSDE), einem System bestehend aus einer stochastischen Vorwärtsgleichung, die vollständig gekoppelt ist an eine Rückwärtsgleichung. Dabei hat die Rückwärtskomponente quadratisches Wachtum in der Kontrollvariablen. Diese Charakterisierung basiert auf Verifikationsargumenten und verwendet Methoden, wie sie in den Anwendungen des Stochastischen Maximumsprinzips oder der Konvexen Dualitätstheorie ebenfalls zutage treten. Im Anschluß zeigen wir die Existenz dieser FBSDE mit Hilfe eines Kompaktheitsresultats für Martingale, die geeignete Normschranken erfüllen. Die Festlegung bestimmter Nutzenfunktionen führt uns schließlich zu einer weiteren Klasse von nicht-standard BSDEs, die in unmittelbarem Zusammenhang zu dem sog. Ansatz der stochastischen partiellen Rückwärts-Differenzialgleichungen (BSPDEs) steht. Im Falle der Exponential-, der Logarithmus- und der Potenznutzenfunktion gilt nämlich eine Trennungseigenschaft für die BSPDEs, die sie in eine deterministische Funktion in ihrer Raumvariablen und einer gewöhnlichen Stochastischen Rückwärts-Differenzialgleichung in ihrer Zeitvariablen separiert.
Anschließend entwickeln wir eine Methode zur numerischen Behandlung von quadratischen BSDEs. Unsere Methode basiert auf einem stochastischen Analogon der Cole-Hopf-Transformation. In einer Anwendung betrachten wir Finanzderivate, die auf illiquide Basiswerte ausgeschrieben sind und zeigen, wie die Verwendung von korrelierten Basiswerten zu Zwecken der Preisbestimmung und des Hedgens heran gezogen werden können. Wir studieren weiterhin eine Klasse von BSDEs, deren Drifts explizite Pfadabhängigkiten aufweisen und leiten mehrere analytische Eigenschaften her. Hierbei ist inbesondere die sog. Pfadregularität zu erwähnen, die im Rahmen von nicht-Markov'schen BSDEs durchaus überraschend ist und das Potenzial birgt, einen implementierbaren numerischen Algorithmus für diese Klasse von BSDEs herzuleiten.
Schließlich studieren wir Dualdarstellungen für Optimalen Mehrfachstoppprobleme. Diese Problemklasse ist anzutreffen z.B. auf Strommärkten, auf dem sich Akteure gegen Preisfluktuationen absichern wollen und hierzu Swing Optionen einsetzen. Der Besitzer einer Swing Option erhält das Recht, zu mehreren Ausübungszeutpunkten eine vertraglich vereinbarte Menge an Strom zu einem Festpreis zu kaufen oder zu verkaufen. Die Bestimmung des “fairen” Preises für solche Derivate bilden den Gegenstand der Optimalen Mehrfachstoppprobleme. Wir leiten Martingal-Dualdarstellungen für die Lösung dieser Probleme her. Diese Darstellungen bilden die Basis für die Entwicklung von Regressions-basierten Monte Carlo Simulationsalgorithmen, die schnell und effektiv untere und obere Preisschranken berechnen. In einer numerischen Studie betrachten wir die Preis-Konfidenzintervalle für eine Swing Option mit Volumenbeschränkungen und Wartezeitrestriktionen.

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Contents

Introduction

The thesis at hand assembles several studies about backward stochastic differential equations (BSDEs) and optimal stopping problems and outlines their applications to the pricing, hedging and securitization of contingent claims on incomplete markets. In particular, the focus is on a small investor who undertakes investments in a financial market, aiming at optimizing her wealth at a prescribed time point in the future. Starting with a given initial capital, she seeks for optimizing this initial wealth by investments into the assets that are given on the market. Characteristic about the small investor is that her transactions do not lead to repercussions on the market itself, i.e. the market dynamics evolve independently. Ubiquitous about financial markets is the presence of random movements and perturbations, thus to achieve the best outcome given the initial capital, the investor has to face her optimization problem under the influence of risks stemming from the stochastic dynamics of the market she acts on.

One of the main challenges is to find a proper way to optimize an objective function subject to stochastic perturbations and constraints and to cast these features into a feasible mathematical framework. In more details, the investor's wealth maximization problem is translated into the mathematical language of stochastic control theory: it offers the a rich enough mathematical formalism that handles optimizing a subjective measurement of wealth and identifying the strategy of how to achieve this goal. However, to make this formalism work in very concrete setups, a number of components have to be implemented first. We mention here that we in particular do not assume that the market is memoryless, i.e. what happens next is independent of what has happened until now. This would naturally lead to the transactions of the investor being independent of the past. We would rather prefer to incorporate a general dependence structure on the market's past perturbations and dynamics which carries over to the transactions of the investor, i.e. the her investment decisions are affected by past events.

The developments of the past twenty years have shown that a powerful and expansive toolkit to deal with financial optimization problems is provided by *backward stochastic differential equations (BSDEs)*. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ and a d-dimensional Brownian motion W, a BSDE is a stochastic differential equation which is typically of the form

$$\mathrm{d}Y_t = -f(t, Y_t, Z_t)\mathrm{d}t + Z_t \mathrm{d}W_t, \;\; Y_T = \xi,$$

where $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a given predictable mapping and ξ is a given $\mathcal{F}_T$-measurable random variable. The task is to find a pair of adapted processes (Y, Z) such that this equation is satisfied a.s. The characteristic feature of a BSDE is the postulation of the terminal condition $Y_T = \xi$. Given the dynamics specified by the *generator* f, the challenge is to steer the evolution of Y into the prescribed terminal state ξ. This feature distinguishes BSDEs from standard forward SDEs. Owing to this, a solution of the BSDE does not only comprise the *value process* Y but also a *martingale control process* Z which caters to the need of correcting drift deviations and navigating the value process into the terminal variable ξ.

The relevance of BSDEs for stochastic optimization problems has been realized first by Bismut [22, 23] who used linear BSDEs to solve stochastic control problems. Another branch where BSDEs are of interest is the domain of Feynman-Kac formulas for non-linear Partial Differential Equations (PDEs). BSDEs turned out to be innately linked to partial differential equations (PDEs) of parabolic and elliptic type. Based on the foundation work by Pardoux and Peng [102], methods were later on refined and tuned into a machinery producing effective and elegant stochastic representations for quasi- and semilinear PDEs of elliptic and parabolic type in Pardoux and Peng [103] and Peng [106]. In fact, the quasilinearity of the PDEs translates into a class of BSDEs whose drifts are of Lipschitz continuous type. This type was the first class of non-linearity that has been studied and is by now coined as *standard* BSDEs.

It was soon realized that BSDEs are a custom-made tool to deal with portfolio optimization problems arising in finance. More concretely, it was realized that the underlying principle for solving the classical Merton problem of portfolio optimization, the *Hamilton-Jacobi-Bellman (HJB)* formalism, could be amplified and elevated into a setting that allows market incompleteness and, more importantly, the incorporation of general past dependence. On the one hand, BSDEs gave a probabilistic alternative of interpreting and solving HJB equations in the Markovian case, and on the other hand, it provided a significant extension of HJB equations to non-Markovian cases. The Merton problem poses the following task to a small investor: given a finite time trading window $[0,T]$, with $T > 0$, the investor can invest into risky assets and riskless bonds. Upon specifying a subjective risk preference, the investor aims at optimizing her expected utility from terminal wealth. This objective can be formulated as the stochastic control problem

$$V(x) := \sup_{\pi} \mathbb{E}[U(x + \int_0^T \pi_u \mathrm{d}S_u)]$$

where $U : \mathbb{R} \to \mathbb{R}$ is a deterministic utility function, S a stochastic process modeling asset prices on a financial market and π the admissible strategies that the investor is allowed to choose. The task is to find an optimal strategy π^* and the optimal expected utility $V(x)$. Abiding to the rules of economic reasoning of what makes a sensible concept of risk, typically concave monotonicity conditions apply for the *value function* $V(x)$. This carries over to their related BSDEs in such a way that 1.) the drifts fall out of the class of Lipschitzianity and exhibit typically a quadratic non-linearity in the control process Z and 2.) the terminal condition and the drift coefficients couple with the dynamics of the underlying market and from a forward-backward SDE (FBSDE). A representative of such an FBSDE which will be encountered later on is

$$
\begin{aligned}
X_t &= x + \int_0^t X_s \left(\frac{1}{1-p} \big(Z_s + \theta_s \big) \right) \mathrm{d}W_s + \int_0^t X_s \left(\frac{1}{1-p} \big(Z_s + \theta_s \big) \theta_s \right) \mathrm{d}s, \\
Y_t &= \log \left(\frac{X_T + H}{X_T} \right)^{p-1} - \int_t^T Z_s \mathrm{d}W_s - \int_t^T \left(\frac{p}{2(p-1)} |Z_s + \theta_s|^2 - \frac{1}{2} |Z_s|^2 \right) \mathrm{d}s,
\end{aligned}
$$

where θ is assumed to be a bounded predictable process and H is assumed to be a sufficiently regular $\mathcal{F}_T$-measurable random variable. The forward process X couples with the control component Z in the drift and the diffusion coefficient in a non-Lipschitz way. The backward process Y has a generator containing terms which are quadratic in Z. The terminal condition couples with the forward process X. We encounter this coupled FBSDE of quadratic growth in the context of utility maximization with respect to the power utility function in Chapter 2. However, it turns out that these coupled FBSDEs with non-Lipschitz generators lead in many cases to an ill-posedness of the equations which makes a solution concept in various regards challenging and more complex than in standard cases.

A self-contained research field, yet solidly connected to numerically solving stochastic control problems, is the branch of Monte Carlo simulation methods for pricing early exercise options pegged to multidimensional underlyings. The conceptual difference to European options is the variability of choosing the exercise time rather than being only allowed to exercise at a prescribed maturity time. Instead of resorting to martingale measure pricing arguments, the pricing of early exercise options is rather implemented by considering its price arising as the value function of an *optimal stopping problem*

$$
V_0 := \sup_{\tau \in [0,T]} \mathbb{E}\big[C_\tau \big],
$$

where $(C_t)_{0 \le t \le T}$ is a sufficiently regular adapted process modeling cashflows and τ is a stopping time taking its values in $[0, T]$. In contrast to the previous optimal control problem, the control π is here replaced with a stopping time τ and the task is to find the optimal stopping time τ^* and the expected value from optimally exercising, V_0, which is thus also the price of the early exercise option. Using stopping times as control variables reflects the feature that the exercise boundary is not fixed but moving. The theoretical backbone for tackling this problem class is the Snell envelope approach: it characterizes the dynamic evolution of the option's value in a backward dynamic programming scheme and establishes (semi-)closed form formulas for the optimal stopping time, see e.g. Föllmer and Schied [52]. With the rise of regression based Monte Carlo methods pioneered by e.g. Carriere [30], Longstaff and Schwartz [85] or Tsitsiklis and Van Roy [124], effective and fast computational techniques have been developed since to evaluated early exercise options, e.g. American options, accurately. By now, it has been realized that algorithms based on the simulation of an approximate optimal stopping time provide lower bounded approximations to the Snell envelope. This approach of tackling directly the optimal stopping time is by now known as the *primal* approach. Based on

Davis and Karatzas [36], Rogers [117] and Haugh and Kogan [55] established an alternative representation for the Snell envelope which completely avoids using stopping times. Instead, one replaces the primal variables stopping times by martingale *dual* variables and represents the Snell envelope by

$$V_0 = \inf_M \mathbb{E}\Big[\sup_{0 \le t \le T} (C_t - M_t)\Big],$$

i.e. the option price arises as an infimum over martingales. This formulation gives rise to numerical methods that approximate the optimal dual martingale. According to the dual representation, solving for dual martingales gives rise to upper bounded approximations the Snell envelope. Therefore, the procedure of computing numerically a good martingale is called the *dual* approach. Putting the focus on computational feasibility and practical relevance, we study optimal stopping problems related to the pricing of contingent claims which are however subject to several exercise rights, hence generalizing the pricing of single early exercise options. A field of particular interest is trading multi-exercise options on electricity markets. In recent years, the deregulation of the energy markets has resulted in higher uncertainties about the short- and intermediate-term development of commodity prices. Taking into account the complex structure of consumption of and the restricted ability to store of electricity, the demand for financial instruments that allow for flexible delivery times as well as a flexible amount of consumption has been rising constantly. One prominent peculiarity that electricity markets juxtapose to other commodity markets is the fact that current cannot be stored. To guarantee the market balance between electricity that is demanded and electricity that is provided, one possibility is to trade on spot markets for electricity. During particular time slots, one or several packages of current can be traded at the price quoted on the spot market. However, these prices are typically governed by a highly oscillatory dynamics with the tendency to exhibit distinctive price peaks, both intraday and over the course of several weeks and months. Thus, from the viewpoint of risk management, features that incorporate protection against price risks are also called for. In this regard, swing options provide their owner with the right to repeatedly buy or sell packages of electricity subject to daily as well as periodic constraints. The number of packages that the owner can buy and sell are generally fixed in advance. Swing options thus equip their owner with the flexibility of delivery and risk protection in a market characterized by price spiking behavior. As mentioned, dual methods allow to bound the price from above while primal methods allow to bound the price from below. Whereas primal methods can be easily adapted to account for general constraints, until recently however, dual methods that allow for e.g. volume constraints and refraction periods, both features existent in real life swing contracts, were not available. Among the literature on dual representations for multi-exercise options, we refer to Meinshausen and Hambly [93] and Schoenmakers [119] and the references therein. To tackle the problem in our setting, we study dual representations for general multiple optimal stopping problems. In particular we include volume constraints and refraction periods simultaneously and we eventually design, discuss and implement a numerical scheme that we apply to swing option pricing.

We can summarize the leitmotif of this thesis as follows: In a first branch, we investigate several classes of non-standard BSDEs which are intimately linked to pricing and hedging problems arising in financial optimization problems. In a second branch, we study generalized multiple stopping problems and establish new dual representations. Making structural assumptions, we furthermore obtain efficient Monte Carlo simulation algorithms for evaluating multi-exercise options. Each chapter is dealing with a (mostly self-contained) own topic and hence can be read as a self-contained unit. The fibres interweaving the chapters and their mutual interrelationship will be pointed out down the road in this presentation. To round up this presentation with an a priori road map, we now give a summary of the content of each chapter in this thesis.

Chapter 1: Forward backward systems for expected utility maximization

This chapter is based on Horst et al. [58]. We study the problem of expected utility maximization from terminal utility under the presence of an endowment. The associated optimal control problem is

$$V(0,x) := \sup_{\pi\in\mathcal{A}} \mathbb{E}[U(X_T^\pi + H)]$$

where U is a real-valued utility function, $\mathcal{A}$ denotes the set of admissible trading strategies, $T < \infty$ is the non-random terminal time, X_T^π is the terminal wealth of the agent arising from the investment strategy $\pi \in \mathcal{A}$, $x > 0$ is the initial capital at time zero and H is an endowment that the agent obtains at terminal time on top of the accumulated wealth X^π. The focus is on the existence and uniqueness of optimal solutions, as well as the characterization of optimal strategy and the *value function* V which is defined for $0 \le t \le T$ and initial wealth $x > 0$ by

$$V(t,x) := \sup_{\pi\in\mathcal{A}} \mathbb{E}\big[U(X_{t,T}^\pi + H)|\mathcal{F}_t\big].$$

Here $X_{t,T}^\pi$ denotes the wealth the agent is able to obtain from trading strategy π during the period $[t,T]$. A powerful and deep reaching technique to tackle the existence of optimal strategies π^* is the *convex dual approach*. It was first proposed by Bismut [23] and later on refined and tuned into a continuous martingale setup by Pliska [112], Karatzas et al. [72, 73], Cvitanic and Karatzas [33]. Its modern and general form within a general semimartingale framework is due to Kramkov and Schachermayer [80]. In these settings, growth conditions on U or related quantities such as the asymptotic elasticity are postulated. Together with mild regularity conditions on the liability and convexity assumptions on the set of admissible trading strategies (see e.g. Biagini [18] for details) they guarantee the existence and uniqueness of optimal investment strategies. Duality techniques, though general and far-reaching, lack however the feature of giving constructive solutions. To our knowledge, up to date, there is no convex duality based numerical implementation of solving utility maximization problems.

An alternative to characterize optimal trading strategies and utilities is provided by forward-backward stochastic differential equations (FBSDE). For exponential utilities it was discussed in El Karoui and Rouge [49] and Sekine [120]. In Hu et al. [62] a general picture was given for classical utility functions without imposing convex constraints on the strategies, but only closed constraints. To paraphrase this approach, assume that the filtration is generated by a standard Wiener process W and assume that $U(x) := -\exp(-\alpha x)$ for some $\alpha > 0$ and $H \in L^2$, or $U(x) := \frac{x^\gamma}{\gamma}$ for $\gamma \in (0,1)$ or $U(x) = \ln x$ and $H = 0$. Let us also assume that admissible strategies are restricted to a closed set. Then, it is shown in Hu et al. [62] that the control problem can be reformulated into solving a BSDE of the form

$$Y_t = H - \int_t^T Z_s \mathrm{d}W_s - \int_t^T f(s, Z_s)\mathrm{d}s, \quad t \in [0,T],$$

where the generator $f(t,z)$ is a predictable process of quadratic growth in the z-variable. We mention that Hu et al. [62] were however only able to make their method work properly in the particular setups of the exponential utility with general endowment, and of the the power or logarithmic utilities with no endowment. In these cases, the portfolio process and the backward component decouple. This approach has been since extended beyond the Brownian framework to more general instances of wealth optimization with complete and incomplete information in various notions, see e.g. Horst et al. [57], Mocha and Westray [95], Morlais [96], Nutz [100] and Mania and Santacroce [90].

In Mania and Tevzadze [92] a verification theorem is derived for optimal trading strategies for more general utility functions in the case $H = 0$. More precisely, given a general utility function U and assuming that there exists an optimal strategy and that the value function exhibits enough regularity in (t,x), it is shown that there exists a predictable random field $(\varphi(t,x))_{(t,x)\in[0,T]\times(0,\infty)}$ such that the pair (V,φ) solves a backward stochastic partial differential equation (BSPDE) of the form

$$V(t,x) = U(x) - \int_t^T \varphi(s,x)\mathrm{d}W_s - \int_t^T \frac{|\varphi_x(s,x)|^2}{V_{xx}(s,x)}\mathrm{d}s, \quad t \in [0,T],$$

where φ_x denotes the partial derivative of φ and V_{xx} the second partial derivative of V with respect to x. The optimal strategy π^* then allows a representation in terms of (V,φ). However, the theory of quadratic growth BSPDE has been developed yet, and to the best of our knowledge the non-linearity arising in the BSPDE cannot be handled unless one considers the classical utility functions where one benefits from the "separation of variables" (see Imkeller et al. [70] or Chapter 3).

What we propose in this chapter is an alternative approach to solve the utility optimization problem for a larger class of utility functions and to characterize the optimal strategy π^* in terms of a fully-coupled system of FBSDEs (instead of a BSPDE). Coupled FBSDEs have been extensively studied in a Lipschitz framework. The treatment has focused mainly on three methods: contraction mappings (Antonelli [7], Pardoux and Tang [104]), a PDE based "4-step scheme" (Ma et al. [89], Delarue [37]), the method of continuation (Hu and Peng [60], Yong [126]). We refer to Ma and Yong [86] for an overview of the general theory on FBSDE with Lipschitz coefficients. The derivation of the FBSDE system appropriate for our purposes starts with a verification type observation. In the case of utility functions defined on $\mathbb{R}$ (if they are defined on $\mathbb{R}_+$, a refinement of the argument will be applicable), given an optimal strategy π^* of the forward portfolio process X^{π^*}, to realize martingale optimality we postulate that $U'(X^{\pi^*}+Y)$ is a martingale, where (Y,Z) is an associated backward process. As a consequence, (Y,Z) is given by a certainty equivalent type expression for the marginal utility $Y=(U')^{-1}(\mathbb{E}(U'(X_T^{\pi^*}+H)|\mathcal{F}_\cdot)-X^{\pi^*}$. This identification allows us to compute the driver of the BSDE related to (Y,Z). It is given in terms of the derivatives of U, involves the optimal forward process X^{π^*}, and provides the backward part of the FBSDE system. In a second step, we consider possible solution triples (X,Y,Z) of the FBSDE system obtained in the first step, not assuming that X corresponds to an optimal portfolio process. We then use a variational perturbation technique well-known in the proofs of the stochastic maximum principle to verify that under some mild conditions on U the triple (X,Y,Z) solves the optimization problem. This in particular means that X coincides with the optimal forward portfolio process X^{π^*}. In summary, under regularity conditions to be specified, solutions (X,Y,Z) of the FBSDE system such that $U'(X+Y)$ is a martingale provide solutions to the original optimization problem. This extends the approach of Hu et al. [62] to the situation of having terminal random endowments by means of a direct translation of the martingale optimality into a system of coupled stochastic equations. However, the challenge is now transferred to solving a fully-coupled FBSDE system (which in general fails to have solutions). However, in classical cases where decoupling techniques apply, our FBSDE system has solutions. Our approach provides in particular an FBSDE system for the case of power utility with general endowments. To the best of our knowledge this is the first treatment that allows to characterize and calculate optimal strategies in this case. We continue the study of this coupled FBSDE in Chapter 2 where we use different approach to construct solutions to the FBSDE. This approach is based on a compactness result for martingales which has been proved by Delbaen and Schachermayer [40].

Chapter 2: Coupled FBSDEs for power utility maximization with endowment

This chapter continues the study of coupled FBSDEs and their relationship to utility maximization problems. In particular, we expose the technical toolkit to solve the special instance of coupled FBSDEs under the power utility function encountered in Chapter 1, i.e. our focus is on the particular FBSDE

$$
\begin{aligned}
X_t &= x+\int_0^t X_s\left(\frac{1}{1-p}(Z_s+\theta_s)\right)\mathrm{d}W_s+\int_0^t X_s\left(\frac{1}{1-p}(Z_s+\theta_s)\theta_s\right)\mathrm{d}s,\\
Y_t &= \log\left(\frac{X_T+H}{X_T}\right)^{p-1}-\int_t^T Z_s\mathrm{d}W_s-\int_t^T\left(\frac{p}{2(p-1)}|Z_s+\theta_s|^2-\frac{1}{2}|Z_s|^2\right)\mathrm{d}s.
\end{aligned}
$$

To this end, we observe that the particular semimartingale transformation $XU'(X)e^Y$ gives rise to a driftless adapted process with a prescribed terminal condition, hence a backward local martingale. We exploit this fact by considering an optimizing sequence for the optimization problem

$$V(x) = \sup_{X_T \in \mathcal{C}(x)} \mathbb{E}\big[U(X_T + H)\big],$$

where $\mathcal{C}(x)$ denotes the admissible strategies with initial wealth $x > 0$, i.e. we assume that there exists a maximizing sequence $(X_T^n)_{n\in\mathbb{N}} \subseteq \mathcal{C}(x)$ such that

$$\lim_{n\to\infty} \mathbb{E}\big[U(X_T^n + H)\big] = V(x).$$

This optimizing sequence $(X_T^n)_{n\in\mathbb{N}}$ is the building block for invoking the celebrated martingale compactness result due to Delbaen and Schachermayer [39]. Paraphrasing this result, given a sequence of martingales $(M^n)_{n\in\mathbb{N}}$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big[\Big(\int_0^T |M_s^n|^2 \mathrm{d}s\Big)^{1/2}\Big] < \infty,$$

i.e. M^n is $\mathcal{H}^1$-bounded, there exists a subsequence $\hat{M}^n$ in the asymptotic convex hull spanned by M^n which converges to a martingale $M \in \mathcal{H}^1$. In conjunction with the martingale transformation $XU'(X)e^Y$, this compactness result produces a limit candidate from which we can reverse engineer the FBSDE components X and (Y, Z).

This approach differs from the usual BSDE approach mentioned in the summary of Chapter 1, because instead of first solving a BSDE by applying the general theory of backward equations, we first solve the optimization problem by means of a convex analysis toolkit for which Komlos type convergence results are paramount. This equips us eventually with an optimal terminal wealth, say X_T^*, to which we need to attach a suitable dynamics such that it falls into the class of admissible terminal wealths $\mathcal{C}(x)$. This is done by employing a celebrated result from stochastic control theory, the *optional decomposition theorem* due to El Karoui and Quenez [48], Kramkov [81]. This result yields that given a semimartingale process S (e.g. the dynamics of the underlying asset market) and given the set of equivalent local martingale measures $\mathbb{Q}$ for S, i.e. probability measures $\mathbb{Q}$ under which S is a local martingale, we consider another process $(D_t)_{0\le t\le T}$ which is a supermartingale under every equivalent martingale measure $\mathbb{Q}$. Then, D_t allows the decomposition

$$D_t = D_0 + \int_0^t \alpha_s \mathrm{d}S_s - C_t$$

where α is integrable with respect to the process S and C is adapted, non-decreasing and $C_0 = 0$. One can check that

$$X_t^* := \operatorname{esssup}_{\mathbb{Q}\in\mathcal{M}^e} \mathbb{E}^Q\big[X_T^* \mid \mathcal{F}_t\big]$$

where $\mathcal{M}^e$ is the set of all martingale measures for S, is a supermartingale under every measure $\mathbb{Q} \in \mathcal{M}^e$, see e.g. Pham [110, Chapter 7]. Invoking optional decomposition for X^* then equips us with an optimal control which steers into the optimal terminal wealth X_T^*.

Hence, we first obtain the optimal terminal wealth and its associated optimal control and then construct the backward equation by utilizing the optimal terminal wealth and the optimal control. An interesting insight is that the identification of the backward equation is realized by means of the stochastic maximum principle. More concretely, the backward equation is the adjoint equation associated to the Hamiltonian system underneath the optimization problem. This workflow reverses the usual BSDE approach which is usually operated in the opposite direction. However, as mentioned before, the BSDE approach fails to launch in the setting of random terminal endowments because one is faced with a coupled FBSDE of quadratic growth which to the best of our knowledge detracts from the available theory on coupled FBSDEs. In this regard, this chapter contributes to the insight that FBSDEs characterizing certain optimization problems can be solved by a toolkit different from the usual BSDE theory approach. Rather, a detour into stochastic convex analysis methods allows for solutions of the FBSDE as a rather straightforward outcome.

Chapter 3: BSDEs related to BSPDEs and applications to utility maximization

This chapter is based on Imkeller et al. [70]. It continues the investigation of the utility maximization problem

$$\sup_{\pi} \mathbb{E}[U(x + \int_0^T \pi_u \mathrm{d}S_u + H)],$$

with $U : \mathbb{R} \to \mathbb{R}$ being a deterministic utility function, S a stochastic process modeling asset prices on a financial market and π the admissible strategies that the investor is allowed to follow. The key topic here is the investigation of the backward stochastic *partial* differential equation (BSPDE) approach to utility maximization as conducted by Mania and Tevzadze [92] and how they are related to ordinary BSDEs. Another issue of interest is to apply computational methods for BSDEs to numerically solve the utility maximization problem. By relating the stochastic control problem to a BSDE, Hu et al. [62] construct in a Brownian setting supermartingales R^π depending on the investor's strategy π such that at maturity, R^π_T coincides with the terminal wealth of the investor. This martingale optimality principle essentially is about solving the utility maximization problem by finding a control π^* such that R^{π^*} is a martingale, hence dominating all other R^π which are supermartingales and thus yields the optimal expected utility. This optimality paradigm has been extended by Musiela and Zariphopoulou [97] and Mania and Tevzadze [92] for general utility functions where the authors characterize the optimal solution of utility maximization problem by a non-linear backward stochastic *partial* differential equation (BSPDE). Within the Brownian framework, BSPDEs with non-linear generators have been studied in Hu and Peng [59] and Peng [105] (see also Hu et al. [61] and the reference therein). In these works, existence and uniqueness results for BSPDEs with generators of quasi- and semilinear type are established. However, the BSPDE from Mania and Tevzadze [92] (even in a Brownian setting) fails to fall into the class of non-linearity under consideration in Hu and Peng [59], Peng [105] and Hu et al. [61]. In Musiela and Zariphopoulou [97] and Mania and Tevzadze [92], existence and uniqueness issues of these non-linear BSPDEs are not dealt with and to the best of our knowledge, no existence (and uniqueness) results seem to exist as of now. However, Mania and Tevzadze [92] show that for classical utility functions (exponential, power and logarithmic) these non-linear BSPDEs reduce to a quadratic growth BSDE with a generator exhibiting a fraction that contains a denominator term in Y which is typically of the form z^2/y. This type of quadratic BSDEs is beyond the limits of the usual requirements for quadratic BSDEs as usually found in Kobylanski [77] that ensure existence or/and uniqueness.

In this chapter, we consider the BSDEs obtained from the BSPDEs from Mania and Tevzadze [92]. We propose a two step reduction algorithm to transform them into coordinates in which existence and uniqueness results are available, and in which they are ultimately accessible for numerical approximation schemes. In a first step, we systematically employ the method of *logarithmic change* of variables to establish existence and uniqueness results. This change of variables was previously used in Hyndman [64] to solve a coupled FBSDE in a particular case, see also Kobylanski [77] and Morlais [96] which use this coordinate change to linearize quadratic growth BSDEs as a stepping stone to prove general existence and uniqueness results. This way, we are able to reduce these BSDEs by a one-to-one map to *standard* quadratic BSDEs, for which many results and tools are available. In a second step, within a predictable representation framework, we provide another one-to-one map which relates this new quadratic growth BSDEs to linear ones. This technique has been employed in Zariphopoulou [128] under the term *distortion* transformation on the level of PDEs to linearize an Hamilton-Jacobi-Bellman equation.

After carrying out this two step algorithm, the stage is set for a numerical solution of the BSDEs from Mania and Tevzadze [92], and thus also for their corresponding portfolio optimization problems. The numerical treatment of such quadratic BSDEs has been realized, depending on the various stages of regularity assumptions on the coefficients, in dos Reis [44], Imkeller and Dos Reis [66] and Richou [116]. Another method for numerically solving quadratic BSDEs can be found in Imkeller et al. [69] which is also part of this thesis in Chapter 4. It employs a method which transforms quadratic BSDEs into BSDEs with Lipschitz continuous drivers, and as such amenable to Monte Carlo schemes as investigated in Bender and Denk [13], Bouchard and Touzi [24], Gobet et al. [54]. This feature of numerical realizability provides an important and practically relevant complement to the theoretical results obtained in the first part of this chapter.

Chapter 4: A Cole-Hopf transformation for quadratic FBSDEs

This chapter is based on Imkeller et al. [69]. The focus of this chapter is on numerics for BSDEs of quadratic growth. In particular, we provide a change of coordinate which resembles the Cole-Hopf transformation from PDE theory which linearizes the Burgers' equation to the heat equation. In our setting, assuming regularity conditions on the coefficients, we analyze a transformation which linearizes the quadratic term in the control component of the generator. This eventually can be exploited for Monte Carlo simulation procedures to simulate solutions of control problems.

Much has been done in recent years to create schemes for BSDEs with Lipschitz continuous drivers (see e.g. Bouchard and Touzi [24] or Elie [51] and references therein). The numerical approximation of BSDEs with drivers of quadratic growth (qgBSDEs) or systems of forward-backward stochastic equations (qgFBSDEs) turns out to be more complicated. In dos Reis [44], a main obstacle was overcome. Following Bouchard and Touzi [24] in the setting of Lipschitz drivers, dos Reis [44] combines two ingredients to prove convergence of a numerical approximation: regularity of the trajectories of the control component of a solution pair of the BSDE in the L^2-sense, a tool first investigated in the framework of Lipschitz continuous BSDEs in Zhang [129], and convenient a priori estimates for the solution. The main difficulty treated in dos Reis [44] consists of establishing path regularity for the control component of the solution pair of the qgBSDE. Then, the quadratic growth part of the driver is truncated to create a sequence of approximating BSDE with Lipschitz continuous drivers. Path regularity is exploited to explicitly capture the convergence rate for the solutions of the truncated BSDE as a function of the truncation height.

An alternative route to avoid the difficulties related to drivers of quadratic growth, and to fall back into the setting of globally Lipschitz ones, consists of using a coordinate transform resembling the celebrated Cole-Hopf transformation from PDE theory. The transformation eliminates the quadratic growth of the driver in the control component at the cost of producing a transformed driver of a new BSDE which in general lacks global Lipschitz continuity. This difficulty can be avoided by some structural hypotheses on the driver. Once such assumptions are postulated, the transformed BSDE enjoys global Lipschitz continuity properties. Therefore the problem of numerical approximation can be tackled in the framework of transformed coordinates by schemes designed for Lipschitz BSDEs. As stated before, this again requires path regularity results in the L^2-sense for the control component of the solution pair of the transformed BSDE. For globally Lipschitz continuous drivers Zhang [129] provides path regularity under simple and mild additional assumptions such as $\frac{1}{2}$-Hölder continuity of the driver in the time variable. The smoothness of the Cole-Hopf transformation allows passing back to the original coordinates without losing path regularity. In summary, if one accepts the additional structural assumptions on the driver, this transformation approach provides numerical approximation schemes for qgBSDE under weaker smoothness conditions for the driver. As an important application of qgBSDE, we deal with the theoretical and numerical description of pricing and hedging contingent claims in incomplete markets. Following Ankirchner et al. [6] and Frei [53], we motivate qgBSDEs by reviewing a simple exponential utility optimization problem resulting from a method to determine the utility indifference price of an insurance related asset in a stylized incomplete market. The setting of the problem allows in particular the reformulation of the problem in terms of a qgBSDE. We illustrate our method by several numerical simulations.

Chapter 5: FBSDEs with time delayed generators

This chapter is based on dos Reis et al. [45]. It is devoted to the investigation of BSDEs with drivers which incorporate functional dependencies of the past. We analyze their solvability in L^p spaces and provide path regularity results that have the potential to pave the way for designing numerical schemes for such BSDEs. The picture we are about to face is the following: by moving away from the usual Markovian setting, i.e. generators that are of the form $f(t, Y_t, Z_t)$ where the value process Y and the control process Z only enter into the generator at the time instance t but not at time instances before t, Delong and Imkeller [42, 43] introduce a new class of BSDE labeled *backward stochastic differential equations with time delayed generators* (delay BSDEs). The dynamics of this class is governed by

$$Y_t = \xi + \int_t^T f(s, Y(s), Z(s))\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \quad t \in [0, T],$$

where the generator f at time $s \in [0, T]$ is allowed to depend on the past values of the solution (Y, Z) over the time interval $[0, s]$ and ξ is an $\mathcal{F}_T$-measurable terminal variable. In these two works the authors show several fundamental properties: existence and uniqueness of a square integrable solution, comparison principles, existence of a measure solution, BMO martingale properties for the control component Z of the solution, Malliavin differentiability for delay BSDEs driven by a Wiener process and a generalized Poisson martingale. To the best of our knowledge the only existence and uniqueness results for this class of BSDEs follow from these two works. As pointed out by Delong [41], delay BSDEs appear naturally in finance related problems about the pricing and hedging of contingent claims. In the same work the author analyzes a vast scope of contracts this class of BSDEs can be applied to.

Paying consideration to and seeking reference from the state of the art of BSDEs with non-time delayed generators, the next step to do for delay BSDEs is to obtain a feasible numerical scheme. Here, the main obstacle is the presence of the control process Z in the generator. This process is usually obtained using the predictable representation property of the underlying stochastic basis, and initially the only known property about Z is that it is a square integrable process. To steer in the direction of a numerical scheme a deeper analysis of the fine properties of the solution of such equations is required. As for numerics for Lipschitz continuous BSDEs (see for example Bouchard and Touzi [24] or Bender and Zhang [15]) one is usually forced to gather several results concerning the *path regularity* properties of the solution process before being able to give proper convergence results. Such path properties include not only sample path continuity but also estimations on the time increments of the components of the solution by the size of the time increment. For the purpose of establishing such path properties we first need to prove several auxiliary results.

Our agenda consists of refining and extending the existence and uniqueness results obtained in Delong and Imkeller [42, 43] and then steer into the direction of the smoothness properties of the solution of delay BSDEs. We start by improving the original results of Delong and Imkeller [42] concerning their a priori estimates by reformulating them in a more standard fashion. In Lemma 2.1 from Delong and Imkeller [42], the a priori estimates on the (norm) difference of the solution of two delay BSDE are written in terms of the difference of the respective terminal conditions and generators. These a priori estimates fall short of the usual a priori estimates one expects to see due to the presence of the solutions of *both* delay BSDE on the right hand side of the estimate. We establish a priori estimates in the classical form where the right hand side of the estimate contains the difference of generators evaluated at their zero spatial state and hence is independent of the BSDE solutions. Within the topic of a priori estimates we extend the results of Delong and Imkeller [42] in another direction. We show that given extra integrability of the terminal condition and the generator, the solution will inherit this integrability. This allows us to state moment and a priori estimates in general L^p-spaces and not solely in L^2. The proof of these estimates relies on techniques from Delong and Imkeller [42] and on computations carried out for non-time delayed BSDEs in the spirit of Wang et al. [125]. The usual techniques to obtain higher order moment estimates fail in the setting of delay BSDEs however. A rough explanation would be that for the usual (non-delay) BSDE setting the dynamics for Y is given by sums of Lebesgue and Itô integrals over the interval $[t,T]$. However, for delay BSDEs the dynamics for Y depends also on a integral over the whole interval $[0,T]$ which eschews the application of the usual a priori estimate techniques. The general estimates we obtain pave the way to a result of existence and uniqueness of solutions for delay BSDEs with Lipschitz continuous generators in general L^p spaces for $p \geq 2$. Inevitably, in analogy to Delong and Imkeller [42, 43] a compatibility condition on the Lipschitz constant and terminal time is required to obtain existence of solutions, see Theorem 5.2.2.

A customary field of application of BSDEs consists in coupling them with SDEs, giving rise (in our case) to systems of delay forward-backward SDEs (delay FBSDEs). We show that when coupling a delay BSDE with a forward diffusion and assuming appropriate regularity conditions, we obtain smoothness properties of the solution in terms of the involved parameters, in particular with respect to the initial condition of the forward diffusion. Combining this with the Malliavin differentiability proved in Delong and Imkeller [43] enables us to derive the usual representation formulas for FBSDE which display the relationship between the Malliavin derivatives of the solution process and their variational derivatives. It is somewhat surprising that such a relationship still holds since this is usually a consequence of the Markov property of BSDEs which clearly fails to materialize in the context of delay FBSDE.

With this collection of results we are finally able to address the path regularity issue for delay BSDE. Using the techniques employed in Imkeller and Dos Reis [66, 67], we establish path continuity for the components of the solution of delay FBSDEs and we give a result that bounds the norm of the increments in time of Y and Z by the size of the time increment. We expect that these results will open the door to the derivation of concrete numerical schemes and their convergence rate and intend to tackle these problems in our future research.

Chapter 6: Dual representations for general multiple stopping problems

This chapter is based on Bender et al. [16]. It is devoted to *dual* representations of multiple optimal stopping problems in a discrete time framework. It takes up the works by Rogers [117] and Haugh and Kogan [55] which derive a "dual" representation for the optimal stopping problem corresponding to the American option pricing problem which is a *single* optimal stopping problem. In their representation the option price is expressed as an infimum of an expectation over a set of martingales. Indeed, the key idea behind this dual representation goes back to Davis and Karatzas [36]. An important issue is to distill from the theoretical insights efficiently working numerical schemes for the pricing of multi-exercise options. For primal methods it is imperative to find good enough approximations to the optimal stopping time (which is explicitly known). In the dual problem however, stopping times are replaced by martingales and approximations hereof eventually give rise to an upper bound for the price of an American option, see e.g. Andersen and Broadie [2] for a realization of the dual approach for calculating upper bounds for American options.

With the emerging importance of electricity and energy markets products with several exercise opportunities, e.g. swing options for buying or selling electricity, the need to price such products have also stimulated activities in extending existing dual representations. In analogy to the pricing of an American option, the pricing of a multi-exercise product leads to a multiple stopping problem, and several numerical methods for solving multiple stopping problems have been proposed since. Let us mention that generalizing existing primal regression methods that seek for good approximations of the optimal stopping times was straightforwardly done by generalizing the approach from Longstaff and Schwartz [85] and Tsitsiklis and Van Roy [124] and by now have become standard. Further, Bender and Schoenmakers [14] developed a kind of policy iteration for multiple stopping. However, the situation for dual representations is more complex. Meinshausen and Hambly [93] proposed a dual representation for the multiple stopping problem by representing the marginal value due to putting an additional right on the top of existing rights as an infimum over a set of martingales and a set of stopping times. This line of research which incorporates martingales and stopping times was continued by Aleksandrov and Hambly [1] and Bender [12] for multi-exercise options under volume constraints, i.e. exercising multiply at a single time instance is allowed up to the limit posed by the volume constraint. In contrast to the dual representation for the marginal value of an additional right, Schoenmakers [119] introduced a dual representation for the aggregated price of a multi-exercise option. The novelty of this new dual representation is that stopping times can be dispensed with and only martingales are needed in the representation. It can thus be considered as the natural extension of the dual representation for single exercise options due to Rogers [117] and Haugh and Kogan [55]. This approach has been carried out further by Bender [11] to a continuous time setting involving (constant) refraction periods.

We mention that Kobylanski et al. [78] introduced and studied multiple stopping problems in the primal fashion in a far more general context, where the payoff is an abstract functional of some ordered sequence of stopping times. The achievement of this chapter is to provide a pure martingale dual representation for such generalized multiple stopping problems in a discrete time setting. For a practical implementation these general representations however gain their full strength of efficiency only if additional regularity assumptions are inducted. In particular, this includes more specifically structured cashflows. Thus, we study generic payoff profiles with both multiplicative and additive structures which incorporate integer valued volume constraints and refraction periods given by stopping times. We moreover design an explicit Monte Carlo algorithm and give a detailed numerical study, exemplifying the pricing of swing options. Compared to benchmark examples, the numerical experiments reveal that our dual algorithms applied to the same class of problems considered in Aleksandrov and Hambly [1] and Bender [12] produce tighter upper bounds on the option price, in particular when the number of exercise rights is large. We moreover present a numerical example which involves swing options subject to both volume constraints and refraction periods and give tight confidence intervals for their respective option prices. This achievement of incorporating volume constraints and refraction periods provides a novelty since to the best of our knowledge, it cannot be treated by the existing dual methods so far.

1 Forward-backward systems for expected utility maximization

In this chapter, we study the utility maximization problem with a general utility function and a random terminal endowment. We develop an approach in which this problem is reduced to the study of a fully-coupled forward backward stochastic differential equation (FBSDE). Assuming the parameters of the problem to be sufficiently regular, we derive the solution to the optimization problem in terms of a forward equation which models the evolution of the optimal wealth process and a backward equation which is fully coupled with the forward component.

1.1 Preliminaries

We consider a financial market which consists of a riskless bond S^0 which we assume here to be of interest rate zero and of $d \geq 1$ stocks given by

$$\mathrm{d}\tilde{S}_t^i := \tilde{S}_t^i \mathrm{d}W_t^i + \tilde{S}_t^i \theta_t^i \mathrm{d}t, \quad i \in \{1, \ldots, d\},$$

where W is a standard Brownian motion on $\mathbb{R}^d$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, $(\mathcal{F}_t)_{t\in[0,T]}$ is the filtration generated by W, and $\theta := (\theta^1, \ldots, \theta^d)$ is the *market price of risk*, a predictable bounded process with values in $\mathbb{R}^d$. Let us remark at this point that a more general framework including a volatility process σ such that $\sigma\sigma^*$ is uniformly elliptic would be straightforwardly possible (see e.g. Hu et al. [62]). But since the main aim of this chapter is to gain some fundamental insights into the link between utility maximization and FBSDEs, we dispense with it here. According to the classical literature (e.g. Delbaen and Schachermayer [40]), in order to exclude arbitrage we assume that the set of equivalent local martingale measures (i.e. probability measures under which $\tilde{S}$ is a local martingale) is not empty. For the sake of simplicity we write throughout this chapter

$$\mathrm{d}S_t^i := \frac{\mathrm{d}\tilde{S}_t^i}{\tilde{S}_t^i}.$$

We denote by $\alpha \cdot \beta$ the inner product in $\mathbb{R}^d$ of vectors α and β and by $|\alpha|$ the usual L^2-norm of a vector $\alpha \in \mathbb{R}^d$. Throughout this chapter, $C > 0$ denotes a generic constant which can differ from line to line. We also define the following spaces:

$$\mathbb{S}^2(\mathbb{R}^d) := \left\{\beta : \Omega \times [0,T] \longrightarrow \mathbb{R}^d : \beta \text{ is predictable and } \mathbb{E}[\sup_{t\in[0,T]} |\beta_t|^2] < \infty\right\},$$

$$\mathbb{H}^2(\mathbb{R}^d) := \left\{\beta : \Omega \times [0,T] \longrightarrow \mathbb{R}^d : \beta \text{ is predictable and } \mathbb{E}\left[\int_0^T |\beta_t|^2 \mathrm{d}t\right] < \infty\right\}.$$

Since the market price of risk θ is assumed to be bounded, the stochastic process

$$\mathcal{E}(-\theta\cdot W)_t := \exp\left(-\int_0^t \theta_s \mathrm{d}W_s - \frac{1}{2}\int_0^t |\theta_s|^2 \mathrm{d}s\right), \quad t \in [0,T],$$

has finite moments of order p for any $p > 0$. We assume $d_1 + d_2 = d$ and that the agent can invest in the assets $\tilde{S}^1, \ldots, \tilde{S}^{d_1}$ while the stocks $\tilde{S}^{d_1+1}, \ldots, \tilde{S}^{d_2}$ cannot be invested into. Denote $S^{\mathcal{H}} := (S^1, \ldots, S^{d_1}, 0 \ldots, 0)$, $W^{\mathcal{H}} := (W^1, \ldots, W^{d_1}, 0 \ldots, 0)$, $W^{\mathcal{O}} := (0, \ldots, 0, W^{d_1+1}, \ldots, W^{d_2})$, and $\theta^{\mathcal{H}} := (\theta^1, \ldots, \theta^{d_1}, 0 \ldots, 0)$ (the superscript $\mathcal{H}$ refers to "hedgeable" and $\mathcal{O}$ to "orthogonal").

The wealth process X^π is defined as

$$X_t^\pi := x + \int_0^t \pi_r \mathrm{d}S_r^{\mathcal{H}} = x + \sum_{i=1}^{d_1} \int_0^t \pi_r^i \mathrm{d}S_r^i, \quad t \in [0,T],$$

where π is an admissible strategy associated with the initial capital $x > 0$ and belongs to the set

$$\Pi^x := \left\{\pi : \Omega \times [0,T] \to \mathbb{R}^{d_1} \text{ predictable: } \mathbb{E}\left[\int_0^T |\pi_t|^2 \mathrm{d}t\right] < \infty, \pi \text{ is self-financing }\right\}. \quad (1.1)$$

Every π in Π^x is extended to an $\mathbb{R}^d$-valued process by concatenating zeros via

$$\tilde{\pi} := (\pi^1, \ldots, \pi^{d_1}, 0, \ldots, 0).$$

In the following, we will always write π in place of $\tilde{\pi}$, i.e. π is an $\mathbb{R}^d$-valued process where the last d_2 components are zero.

Moreover, we consider a utility function $U : I \to \mathbb{R}$ where I is an interval on $\mathbb{R}$ such that U is strictly increasing and strictly concave. We seek for a strategy $\pi^* \in \Pi^x$ satisfying $\mathbb{E}[U(X_T^{\pi^*} + H)] < \infty$ such that

$$\pi^* = \mathrm{argmax}_{\pi\in\Pi^x} \mathbb{E}\Big[U(X_T^\pi + H)\Big], \quad (1.2)$$

where H is a random variable in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$. In the subsequent sections we will elaborate on the necessary and sufficient conditions such that (1.2) becomes well-posed.

1.2 Utilities defined on the real line

In this section we consider a utility function $U:\mathbb{R}\to\mathbb{R}$ defined on the whole real line. We assume that U is strictly increasing and strictly concave and that the investor is equipped with an endowment $H\in L^2(\Omega,\mathcal{F}_T,\mathbb{P})$ at the terminal time T. We assume the following conditions to hold:

(H1) $U:\mathbb{R}\to\mathbb{R}$ is three times continuously differentiable.

(H2) We say that condition (H2) holds for an element $\pi^*\in\Pi^x$ if $\mathbb{E}\big[U'(X_T^{\pi^*}+H)^2\big]<\infty$ and if for every bounded predictable real-valued process $(h_t)_{t\in[0,T]}$, the family of random variables

$$\left(\int_0^T h_u \mathrm{d}S_u^{\mathcal{H}} \int_0^1 U'\left(X_T^{\pi^*}+H+\varepsilon\theta\int_0^T h_r\mathrm{d}S_r^{\mathcal{H}}\right)\mathrm{d}\theta\right)_{\varepsilon\in(0,1)}$$

is uniformly integrable.

Before presenting the first main result of this section, we check that condition **(H2)** is satisfied for every strategy π^* such that $\mathbb{E}[|U'(X_T^{\pi^*}+H)|]<\infty$, given that the following exponential growth condition on the marginal utility is satisfied:

$$U'(x+y)\le C\left(1+U'(x)\right)(1+\exp(\alpha y))\quad \text{for some } \alpha\in\mathbb{R}.$$

Indeed, let $G:=\int_0^T h_r\mathrm{d}S_r^{\mathcal{H}}$ and $d>0$. We show that

$$q(d):=\sup_{\varepsilon\in(0,1)}\mathbb{E}\left[\left|G\int_0^1 U'(X_T^{\pi^*}+H+\varepsilon rG)dr\right|\mathbf{1}_{\left|G\int_0^1 U'(X_T^{\pi^*}+H+\varepsilon rG)\mathrm{d}r\right|>d}\right]$$

vanishes as $d\to\infty$. For simplicity we denote

$$\delta_{\varepsilon,d}:=\mathbf{1}_{\left|G\int_0^1 U'(X_T^{\pi^*}+H+\varepsilon rG)\mathrm{d}r\right|>d}.$$

By the Cauchy-Schwarz inequality we have

$$\begin{aligned} q(d)&\le\sup_{\varepsilon\in(0,1)}\mathbb{E}\left[(1+U'(X_T^{\pi^*}+H))\left|G(1+\int_0^1\exp(\alpha\varepsilon rG))\mathrm{d}r\right|\delta_{\varepsilon,d}\right]\\ &\le C\mathbb{E}\left[|U'(X_T^{\pi^*}+H)|^2\right]^{1/2}\sup_{\varepsilon\in(0,1)}\mathbb{E}\left[\left|G\int_0^1\exp(\alpha\varepsilon rG)\mathrm{d}r\right|^2\delta_{\varepsilon,d}\right]^{1/2}. \end{aligned}$$

Since $\mathbb{E}\left[U'(X_T^{\pi^*}+H)^2\right]$ is assumed to be finite we deduce from the inequality

$$\exp(\alpha\zeta x)\le 1+\exp(\alpha x)\quad\text{for all } x\in\mathbb{R},\ 0<\zeta<1,$$

that

$$q(d) \le C \sup_{\varepsilon\in(0,1)} \mathbb{E}\left[|G(2+\exp(\alpha G))|^2 \delta_{\varepsilon,d}\right]^{1/2}.$$

Applying successively the Cauchy-Schwarz inequality and the Markov inequality, we see that

$$\begin{aligned} q(d) &\le C\mathbb{E}\left[|G(2+\exp(\alpha G))|^4\right]^{1/4} \sup_{\varepsilon\in(0,1)} \mathbb{E}[\delta_{\varepsilon,d}]^{1/4} \\ &\le C\mathbb{E}\left[|G(2+\exp(\alpha G))|^4\right]^{1/4} d^{-1/4} \sup_{\varepsilon\in(0,1)} \mathbb{E}\left[|G| \int_0^1 U'(X_T^{\pi^*} + H + \varepsilon r G)\mathrm{d}r\right]^{1/4} \\ &\le C\mathbb{E}\left[|G(2+\exp(\alpha G))|^4\right]^{1/4} d^{-1/4}\, \mathbb{E}\left[|G(2+\exp(\alpha G))|^2\right]^{1/8}. \end{aligned}$$

Let $p \ge 2$. Since h and θ are bounded it is clear that $\mathbb{E}\left[|G|^{2p}\right] < \infty$ and

$$\begin{aligned} &\mathbb{E}\left[|G(2+\exp(\alpha G))|^p\right] \\ &\le \mathbb{E}\left[|G|^{2p}\right]^{1/2} \mathbb{E}\left[|2+\exp(\alpha G)|^{2p}\right]^{1/2} \\ &\le C\left(2+\mathbb{E}\left[|\exp(\alpha G)|^{2p}\right]\right)^{1/2} \\ &= C\Bigg(2+\mathbb{E}\Bigg[\exp\left(\int_0^T 2p\alpha h_r dW_r^{\mathcal{H}} - \frac{1}{2}\int_0^T |2p\alpha h_r|^2 \mathrm{d}r\right) \\ &\qquad \exp\left(\frac{1}{2}\int_0^T |2p\alpha h_r|^2 + 2p\alpha h_r\cdot\theta_r \mathrm{d}r\right)\Bigg]\Bigg)^{1/2} \\ &\le C. \end{aligned}$$

Hence, $\lim_{d\to\infty} q(d) = 0$ which proves the claim.

1.2.1 Characterization and verification: incomplete markets

We now present a first main result of this chapter, a verification theorem for optimal trading strategies.

Theorem 1.2.1. *Assume that **(H1)** holds. Let $\pi^* \in \Pi^x$ be an optimal solution to the problem* (1.2) *which satisfies **(H2)**. Then there exists a predictable process $(Y_t)_{t\in[0,T]}$ with $Y_T = H$ such that $U'(X^{\pi^*}+Y)$ is a square integrable martingale. Moreover, the optimal strategy allows for the representation*

$$\pi_t^{*^i} = -\theta_t^i \frac{U'(X_t^{\pi^*}+Y_t)}{U''(X_t^{\pi^*}+Y_t)} - Z_t^i, \quad t\in[0,T], \quad i=1,\dots,d_1,$$

where we have $Z_t := \frac{\mathrm{d}\langle Y,W\rangle_t}{\mathrm{d}t} := \left(\frac{\mathrm{d}\langle Y,W^1\rangle_t}{\mathrm{d}t},\dots,\frac{\mathrm{d}\langle Y,W^d\rangle_t}{\mathrm{d}t}\right)$.

Proof. We first prove the existence of Y. Since $\mathbb{E}\left[U'(X_T^{\pi^*}+H)^2\right] < \infty$, the process α

defined as $\alpha_t := \mathbb{E}\big[U'(X_T^{\pi^*} + H)|\mathcal{F}_t\big]$, for $t \in [0,T]$, is a square integrable martingale. Define $Y_t := (U')^{-1}(\alpha_t) - X_t^{\pi^*}$. Then Y is $\mathcal{F}_t$-predictable. Applying Itô's formula yields

$$Y_t + X_t^{\pi^*} = Y_T + X_T^{\pi^*} - \int_t^T \frac{1}{U''(U'^{-1}(\alpha_t))}\mathrm{d}\alpha_t + \frac{1}{2}\int_t^T \frac{U^{(3)}(U'^{-1}(\alpha_t))}{(U''(U'^{-1}(\alpha_t)))^3}\mathrm{d}\langle\alpha,\alpha\rangle_t. \quad (1.3)$$

We also note that α is the unique solution of the zero driver BSDE

$$\alpha_t = U'(X_T^{\pi^*} + Y_T) - \int_t^T \beta_s \mathrm{d}W_s, \quad t \in [0,T], \quad (1.4)$$

where β is a square integrable predictable process with values in $\mathbb{R}^d$. Plugging (1.4) into (1.3) yields

$$Y_t + X_t^{\pi^*} = X_T^{\pi^*} + H - \int_t^T \frac{1}{U''(X_s^{\pi^*} + Y_s))}\beta_s \mathrm{d}W_s + \frac{1}{2}\int_t^T \frac{U^{(3)}(X_s^{\pi^*} + Y_s)}{(U''(X_s^{\pi^*} + Y_s))^3}|\beta_s|^2\mathrm{d}s.$$

Setting $\tilde{Z} := \frac{1}{U''(X^{\pi^*}+Y)}\beta$, we have

$$Y_t + X_t^{\pi^*} = X_T^{\pi^*} + H - \int_t^T \tilde{Z}_s \mathrm{d}W_s + \frac{1}{2}\int_t^T \frac{U^{(3)}}{U''}(X_s^{\pi^*} + Y_s)|\tilde{Z}_s|^2\mathrm{d}s.$$

Now by putting $Z^i := \tilde{Z}^i - \pi^{*i}$ for $i = 1,\dots,d$, we conclude that Y is a solution to the BSDE

$$Y_t = H - \int_t^T Z_s \mathrm{d}W_s - \int_t^T f(s, X_s^{\pi^*}, Y_s, Z_s)\mathrm{d}s, \quad t \in [0,T], \quad (1.5)$$

where the driver f is given by

$$f(s, X_s^{\pi^*}, Y_s, Z_s) = -\frac{1}{2}\frac{U^{(3)}}{U''}(X_s^{\pi^*} + Y_s)|\pi_s^* + Z_s|^2 - \pi_s^* \cdot \theta_s. \quad (1.6)$$

Finally, by construction we have $U'(X_t^{\pi^*} + Y_t) = \alpha_t$, thus it is a martingale. Now let us deal with the representation of the optimal strategy. To this end, let $h : [0,T] \times \Omega \to \mathbb{R}^{d_1}$ be a bounded predictable process. We extend h into $\mathbb{R}^d$ by concatenating zeros via $\tilde{h} := (h^1,\dots,h^{d_1},0,\dots,0)$ and by abuse of notation denote $\tilde{h}$ again by h. Thus for every $\varepsilon \in (0,1)$ the perturbed strategy $\pi^* + \varepsilon h$ is again an element from Π^x. By the optimality of π^*, it is clear that for every such h we have

$$l(h) := \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}\Big[U(x + \int_0^T (\pi_r^* + \varepsilon h_r)\mathrm{d}S_r^{\mathcal{H}} + Y_T) - U(x + \int_0^T \pi_r^* \mathrm{d}S_r^{\mathcal{H}} + Y_T)\Big] \le 0. \quad (1.7)$$

Moreover, we have

$$\begin{aligned}&\frac{1}{\varepsilon}\Big(U(x+\int_0^T(\pi_r^*+\varepsilon h_r)\mathrm{d}S_r^{\mathcal{H}}+Y_T)-U(x+\int_0^T\pi_r^*\mathrm{d}S_r^{\mathcal{H}}+Y_T)\Big)\\&\quad=\int_0^T h_r\mathrm{d}S_r^{\mathcal{H}}\int_0^1 U'\Big(X_T^{\pi^*}+Y_T+\theta\varepsilon\int_0^T h_r\mathrm{d}S_r^{\mathcal{H}}\Big)\mathrm{d}\theta.\end{aligned}$$

Using **(H2)**, Lebesgue's dominated convergence theorem implies that (1.7) can be rewritten as

$$\mathbb{E}\Big[U'(X_T^{\pi^*}+Y_T)\int_0^T h_r\mathrm{d}S_r^{\mathcal{H}}\Big]\le 0 \tag{1.8}$$

for every bounded predictable process h. Applying the integration by parts formula to $U'(X_s^{\pi^*}+Y_s)_{s\in[0,T]}$ and $\big(\int_0^s h_r\mathrm{d}S_r^{\mathcal{H}}\big)_{s\in[0,T]}$, we get

$$\begin{aligned}&U'(X_T^{\pi^*}+Y_T)\int_0^T h_r\mathrm{d}S_r^{\mathcal{H}}\\&\quad=U'(x+Y_0)\times 0+\int_0^T U'(X_s^{\pi^*}+Y_s)h_s\mathrm{d}S_s^{\mathcal{H}}\\&\qquad+\int_0^T\int_0^s h_r\mathrm{d}S_r^{\mathcal{H}}\,U''(X_s^{\pi^*}+Y_s)\Big[(\pi_s^*+Z_s)\mathrm{d}W_s^{\mathcal{H}}+(\pi_s^*\cdot\theta_s+f(s,X_s^{\pi^*},Y_s,Z_s))\mathrm{d}s\Big]\\&\qquad+\frac{1}{2}\int_0^T\int_0^s h_r\mathrm{d}S_r^{\mathcal{H}}\,U^{(3)}(X_s^{\pi^*}+Y_s)|\pi_s^*+Z_s|^2\mathrm{d}s\\&\qquad+\int_0^T U''(X_s^{\pi^*}+Y_s)h_s\cdot(\pi_s^*+Z_s)\mathrm{d}s.\end{aligned}$$

Using the expression for f from (1.6), the previous equality transforms into

$$\begin{aligned}&U'(X_T^{\pi^*}+Y_T)\int_0^T h_r\mathrm{d}S_r^{\mathcal{H}}\\&\quad=\int_0^T\Big(U'(X_s^{\pi^*}+Y_s)\theta_r+U''(X_s^{\pi^*}+Y_s)(\pi_s^*+Z_s)\Big)\cdot h_r\mathrm{d}r\\&\qquad+\int_0^T\int_0^s h_r\mathrm{d}S_r^{\mathcal{H}}\,U''(X_s^{\pi^*}+Y_s)(\pi_s^*+Z_s)\mathrm{d}W_s^{\mathcal{H}}+\int_0^T U'(X_s^{\pi^*}+Y_s)h_s\mathrm{d}W_s^{\mathcal{H}}.\end{aligned}\tag{1.9}$$

The next step is to take conditional expectations in (1.9). However the two terms on the second line of the right hand side are a priori only local martingales, hence the task is to check that both expressions exhibit enough integrability to apply conditional expectations. We start by showing that the first term is a uniformly integrable martingale. Indeed, from the computations which lead to (1.5) we have

$$U''(X^{\pi^*}+Y)(\pi^*+Z)=\beta,$$

where β is the square integrable process appearing in (1.4). Using the Burkholder-Davis-

Gundy (BDG) inequality we get

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,T]}\left|\int_0^s\int_0^r h_u \mathrm{d}S_u^{\mathcal{H}}\, U''(X_r^{\pi^*}+Y_r)(\pi_r^*+Z_r)\mathrm{d}W_r^{\mathcal{H}}\right|\right]\\
&\leq C\mathbb{E}\left[\left|\int_0^T\left|\int_0^s h_r \mathrm{d}S_r^{\mathcal{H}}\right|^2|\beta_s|^2\mathrm{d}s\right|^{1/2}\right]\\
&\leq C\mathbb{E}\left[\left(\sup_{s\in[0,T]}\left|\int_0^s h_r\mathrm{d}S_r^{\mathcal{H}}\right|^2\right)^{1/2}\left(\int_0^T|\beta_s|^2\mathrm{d}s\right)^{1/2}\right].
\end{aligned}
$$

An application of Young's inequality furthermore yields

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sup_{s\in[0,T]}\left|\int_0^s h_r\mathrm{d}S_r^{\mathcal{H}}\right|^2\right)^{1/2}\left(\int_0^T|\beta_s|^2\mathrm{d}s\right)^{1/2}\right]\\
&\leq C\mathbb{E}\left[\sup_{s\in[0,T]}\left|\int_0^s h_r\mathrm{d}S_r^{\mathcal{H}}\right|^2\right]+C\mathbb{E}\left[\int_0^T|\beta_s|^2\mathrm{d}s\right]\\
&\leq C\left(1+\mathbb{E}\left[\sup_{s\in[0,T]}\left|\int_0^s h_r\mathrm{d}W_r^{\mathcal{H}}\right|^2\right]\right),
\end{aligned}
$$

where we use the boundedness of h and θ. Applying again the BDG inequality, we obtain

$$
\mathbb{E}\left[\sup_{s\in[0,T]}\left|\int_0^s h_r\mathrm{d}W_r^{\mathcal{H}}\right|^2\right]\leq C\mathbb{E}\left[\int_0^T|h_r|^2\mathrm{d}r\right]<\infty.
$$

Putting together the previous steps, we get

$$
\mathbb{E}\left[\sup_{s\in[0,T]}\left|\int_0^s\int_0^r h_u\mathrm{d}S_u^{\mathcal{H}}\, U''(X_r^{\pi^*}+Y_r)(\pi_r^*+Z_r)\mathrm{d}W_r^{\mathcal{H}}\right|\right]<\infty,
$$

which thus yields

$$
\mathbb{E}\left[\int_0^T\int_0^s h_r dS_r^{\mathcal{H}}\, U''(X_s^{\pi^*}+Y_s)(\pi_s^*+Z_s)\mathrm{d}W_s^{\mathcal{H}}\right]=0.
$$

Note that $\left(\int_0^t U'(X_s^{\pi^*}+Y_s)h_s\mathrm{d}W_s^{\mathcal{H}}\right)_{t\in[0,T]}$ is a square integrable martingale. To see this, recall that $U'(X^{\pi^*}+Y)=\alpha$ is a square integrable martingale and thus

$$
\mathbb{E}\left[\int_0^T\left|U'(X_s^{\pi^*}+Y_s)h_s\right|^2\mathrm{d}s\right]<\infty.
$$

Similarly we have

$$
\mathbb{E}\left[\left|U'(X_T^{\pi^*}+Y_T)\int_t^T h_r\mathrm{d}S_r^{\mathcal{H}}\right|\right]<\infty.
$$

Now we can take expectations in (1.9) and obtain

$$\begin{aligned}
&\mathbb{E}\Big[U'(X_T^{\pi^*}+Y_T)\int_0^T h_r \mathrm{d}S_r^{\mathcal{H}}\Big] \\
&= \mathbb{E}\Big[\int_0^T \Big(U'(X_s^{\pi^*}+Y_s)\theta_r + U''(X_s^{\pi^*}+Y_s)(\pi_s^*+Z_s)\Big)\cdot h_r \mathrm{d}r\Big],
\end{aligned} \tag{1.10}$$

which in conjunction with (1.8) leads to

$$\mathbb{E}\Big[\int_0^T \Big(U'(X_s^{\pi^*}+Y_s)\theta_r + U''(X_s^{\pi^*}+Y_s)(\pi_s^*+Z_s)\Big)\cdot h_r \mathrm{d}r\Big] \le 0$$

for every bounded predictable process h. Replacing h by $-h$, we get

$$\mathbb{E}\Big[\int_0^T \Big(U'(X_s^{\pi^*}+Y_s)\theta_r + U''(X_s^{\pi^*}+Y_s)(\pi_s^*+Z_s)\Big)\cdot h_r \mathrm{d}r\Big] = 0. \tag{1.11}$$

Now fix i in $\{1,\dots,d_1\}$ and denote $A_s^i := U'(X_s^{\pi^*}+Y_s)\theta_r + U''(X_s^{\pi^*}+Y_s)(\pi_s^{*^i}+Z_s^i)$, $h_s := (0,\dots,0,\mathbb{1}_{A_s^i>0},0,\dots,0)$ where the non-vanishing component is in the i-th entry. From (1.11) we get

$$\mathbb{E}\Big[\int_0^T \mathbb{1}_{A_s^i>0}\Big[U'(X_s^{\pi^*}+Y_s)\theta_s^i + U''(X_s^{\pi^*}+Y_s)(\pi_s^{*^i}+Z_s^i)\Big]\mathrm{d}s\Big] = 0,$$

which implies $\mathrm{d}\mathbb{P}\otimes\mathrm{d}t - a.s.$ that $A^i \le 0$. Similarly, by choosing

$$h_s = (0,\dots,0,\mathbb{1}_{A_s^i<0},0,\dots,0)$$

we deduce that

$$U'(X^{\pi^*}+Y)\theta^i + U''(X^{\pi^*}+Y_t)(\pi_t^{*^i}+Z_t^i) = 0, \quad \mathrm{d}\mathbb{P}\otimes\mathrm{d}t - a.s.$$

Since $i\in\{1,\dots,d_1\}$ is arbitrary, this concludes the proof. □

The verification theorem above can also be expressed in terms of a fully coupled forward-backward system.

Theorem 1.2.2. *Under the assumptions of Theorem 1.2.1, the solution π^* of the optimization problem* (1.2) *is given by*

$$\pi_t^{*^i} = -\theta_t^i\frac{U'(X_t+Y_t)}{U''(X_t+Y_t)} - Z_t^i, \quad t\in[0,T], \quad i=1,\dots,d_1,$$

where (X,Y,Z) with values in $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d$ is a triplet of adapted processes which solves *the FBSDE*

$$\begin{cases} X_t = x - \int_0^t \left(\theta_s \frac{U'(X_s+Y_s)}{U''(X_s+Y_s)} + Z_s\right) \mathrm{d}W_s^{\mathcal{H}} - \int_0^t \left(\theta_s \frac{U'(X_s+Y_s)}{U''(X_s+Y_s)} + Z_s\right) \cdot \theta_s^{\mathcal{H}} \mathrm{d}s, \\ Y_t = H - \int_t^T Z_s \mathrm{d}W_s - \int_t^T \Big[-\frac{1}{2} |\theta_s^{\mathcal{H}}|^2 \frac{U^{(3)}(X_s+Y_s)|U'(X_s+Y_s)|^2}{U''(X_s+Y_s)^3} \\ \quad + |\theta_s^{\mathcal{H}}|^2 \frac{U'(X_s+Y_s)}{U''(X_s+Y_s)} + Z_s \cdot \theta_s^{\mathcal{H}} - \frac{1}{2} |Z_s^{\mathcal{O}}|^2 \frac{U^{(3)}}{U''}(X_s+Y_s) \Big] \mathrm{d}s, \end{cases} \tag{1.12}$$

with the notation $Z = (\underbrace{Z^1, \dots, Z^{d_1}}_{=:Z^{\mathcal{H}}}, \underbrace{Z^{d_1+1}, \dots, Z^d}_{=:Z^{\mathcal{O}}})$. *Moreover, the optimal wealth process* X^{π^*} *is equal to* X.

Proof. From Theorem 1.2.1 we know that the optimal strategy is given by

$$\pi_t^{*^i} = -\theta_t^i \frac{U'(X_t^{\pi^*} + Y_t)}{U''(X_t^{\pi^*} + Y_t)} - Z_t^i, \quad t \in [0, T], \quad i = 1, \dots, d_1,$$

where (Y, Z) is a solution to the BSDE (1.5) with the generator f given by (1.6). Now plugging the representation of π^* into (1.6) yields

$$\begin{cases} X_t^{\pi^*} = x - \int_0^t \left(\theta_s \frac{U'(X_s^{\pi^*}+Y_s)}{U''(X_s^{\pi^*}+Y_s)} + Z_s\right) \mathrm{d}W_s^{\mathcal{H}} - \int_0^t \left(\theta_s \frac{U'(X_s^{\pi^*}+Y_s)}{U''(X_s^{\pi^*}+Y_s)} + Z_s\right) \cdot \theta_s^{\mathcal{H}} \mathrm{d}s, \\ Y_t = H - \int_t^T Z_s \mathrm{d}W_s - \int_t^T \Big[-\frac{1}{2} |\theta_s^{\mathcal{H}}|^2 \frac{U^{(3)}(X_s^{\pi^*}+Y_s)|U'(X_s^{\pi^*}+Y_s)|^2}{(U''(X_s^{\pi^*}+Y_s))^3} \\ \quad + |\theta_s^{\mathcal{H}}|^2 \frac{U'(X_s^{\pi^*}+Y_s)}{U''(X_s^{\pi^*}+Y_s)} + Z_s \cdot \theta_s^{\mathcal{H}} - \frac{1}{2} |Z_s^{\mathcal{O}}|^2 \frac{U^{(3)}}{U''}(X_s^{\pi^*}+Y_s) \Big] \mathrm{d}s. \end{cases} \tag{1.13}$$

Recalling that we have $X^{\pi} = x + \int_0^{\cdot} \pi_s (\mathrm{d}W_s^{\mathcal{H}} + \theta_s^{\mathcal{H}} \mathrm{d}s)$ for any admissible strategy π, we get the forward part of the FBSDE. □

Remark 1.2.1. *Note that using Itô's formula and the FBSDE* (1.12)*, we see that*

$$U'(X+Y) = U'(x+Y_0) + \int_0^{\cdot} -\theta_s^{\mathcal{H}} U'(X_s+Y_s) \mathrm{d}W_s^{\mathcal{H}} + \int_0^{\cdot} U''(X_s+Y_s) Z_s^{\mathcal{O}} \mathrm{d}W_s^{\mathcal{O}},$$

i.e. $U'(X+Y)$ *is a local martingale.*

Remark 1.2.2. *Note that using the system* (1.12)*, we obtain for* $\alpha := U'(X^{\pi^*} + Y)$

$$\begin{aligned} & U'(X_t^{\pi^*} + Y_t)(X_t^{\pi} - X_t^{\pi^*}) \\ & = \int_0^t (X_s^{\pi} - X_s^{\pi^*}) \mathrm{d}\alpha_s + \int_0^t \alpha_s (\pi_s - \pi_s^*) \mathrm{d}W_s^{\mathcal{H}} \\ & \quad + \int_0^t \left(\alpha_s \theta_s^{\mathcal{H}} + U''(X_s^{\pi^*} + Y_s)(Z_s^{\mathcal{H}} + \pi_s^*)\right) \cdot (\pi_s - \pi_s^*) \mathrm{d}s \\ & = \int_0^t (X_s^{\pi} - X_s^{\pi^*}) \mathrm{d}\alpha_s + \int_0^t \alpha_s (\pi_s - \pi_s^*) \mathrm{d}W_s^{\mathcal{H}}, \end{aligned}$$

showing that $U'(X^{\pi^*}+Y)(X^\pi - X^{\pi^*})$ *is a local martingale for every* π *in* Π^x.

The converse implication of Theorems 1.2.1 and 1.2.2 constitutes a second main result.

Theorem 1.2.3. *Let **(H1)** hold and let* (X,Y,Z) *be a triplet of predictable processes solving the FBSDE* (1.12) *such that*

- $Z \in \mathbb{H}^2(\mathbb{R}^d)$;
- $\mathbb{E}\big[|U'(X_T+H)|^2\big] < \infty$;
- $U'(X+Y)$ *is a non-negative martingale.*

Moreover, assume that there exists a constant $\kappa > 0$ *such that*

$$-\frac{U'(x)}{U''(x)} \le \kappa \text{ for all } x \in \mathbb{R}.$$

Then

$$\pi_t^{*^i} := -\frac{U'(X_t+Y_t)}{U''(X_t+Y_t)}\theta_t^i - Z_t^i, \quad t\in[0,T], \quad i\in\{1,\dots,d_1\},$$

is a solution to the optimization problem (1.2).

Proof. First, note that due to the definition of π^*, we have $X = X^{\pi^*}$. Since the risk tolerance $-\frac{U'(x)}{U''(x)}$ is bounded and since $Z\in\mathbb{H}^2(\mathbb{R}^d)$, we immediately get $\mathbb{E}\left[\int_0^T|\pi_s^*|^2\mathrm{d}s\right] < \infty$, implying $\pi\in\Pi^x$. By assumption, $U'(X+Y)$ is a non-negative martingale, hence there exists a local martingale L such that $U'(X+Y) = \mathcal{E}(L)$. Applying Itô's formula yields

$$L = \log(U'(x+Y_0)) + \int_0^\cdot -\theta_s^{\mathcal{H}}\mathrm{d}W_s^{\mathcal{H}} + \int_0^\cdot \frac{U''(X_s+Y_s)}{U'(X_s+Y_s)}Z_s^{\mathcal{O}}\mathrm{d}W_s^{\mathcal{O}}.$$

Let us define the probability measure $\mathbb{Q}\sim\mathbb{P}$ by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} := \frac{U'(X_T+H)}{\mathbb{E}[U'(X_T+H)]}.$$

Then, Girsanov's theorem implies that

$$\begin{aligned}\tilde W &:= \tilde W^{\mathcal{H}} + \tilde W^{\mathcal{O}}\\ &= \Big(W^1 + \int\theta^1\mathrm{d}t,\dots,W^{d_1}+\int\theta^{d_1}\mathrm{d}t, W^{d_1+1} - \int\frac{U''(X+Y)}{U'(X+Y)}Z^{d_1+1}\mathrm{d}t,\\ &\qquad \dots, W^{d_2} - \int\frac{U''(X+Y)}{U'(X+Y)}Z^{d_2}\mathrm{d}t\Big)\end{aligned}$$

is a standard Brownian motion under $\mathbb{Q}$. Thus X^π is a local martingale under $\mathbb{Q}$ for every π in Π^x. Now fix π in Π^x. Let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence for the local martingale $X^\pi - X^{\pi^*}$. Since U is a concave, we have

$$U(X_T^\pi+H) - U(X_T^{\pi^*}+H) \le U'(X_T^{\pi^*}+H)(X_T^\pi - X_T^{\pi^*}). \tag{1.14}$$

Taking expectations in (1.14) we get

$$\begin{aligned}\frac{\mathbb{E}[U(X_T^{\pi}+H)-U(X_T^{\pi^*}+H)]}{\mathbb{E}[U'(X_T+H)]} &\leq \mathbb{E}_{\mathbb{Q}}[X_T^{\pi}-X_T^{\pi^*}]\\ &= \mathbb{E}_{\mathbb{Q}}\left[\lim_{n\to\infty}\int_0^{T\wedge\tau_n}(\pi_s-\pi_s^*)\mathrm{d}\tilde{W}_s^{\mathcal{H}}\right]\\ &= \lim_{n\to\infty}\mathbb{E}_{\mathbb{Q}}\left[\int_0^{T\wedge\tau_n}(\pi_s-\pi_s^*)\mathrm{d}\tilde{W}_s^{\mathcal{H}}\right]\\ &= 0,\end{aligned}$$

which follows from Lebesgue's dominated convergence theorem. To see this, we prove

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[0,T]}\left|\int_0^t(\pi_s-\pi_s^*)\mathrm{d}\tilde{W}_s^{\mathcal{H}}\right|\right]<\infty.$$

Indeed the BDG inequality and the Cauchy-Schwarz inequality imply

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[0,T]}\left|\int_0^t(\pi_s-\pi_s^*)\mathrm{d}\tilde{W}_s^{\mathcal{H}}\right|\right]\\ &\leq C\mathbb{E}_{\mathbb{Q}}\left[\left(\int_0^T|\pi_s-\pi_s^*|^2\mathrm{d}s\right)^{\frac{1}{2}}\right]\\ &= C\mathbb{E}\left[\frac{U'(X_T+H)}{\mathbb{E}[U'(X_T+H)]}\left(\int_0^T|\pi_s-\pi_s^*|^2\mathrm{d}s\right)^{\frac{1}{2}}\right]\\ &\leq C\mathbb{E}\left[\left|\frac{U'(X_T+H)}{\mathbb{E}[U'(X_T+H)]}\right|^2\right]^{\frac{1}{2}}\mathbb{E}\left[\int_0^T|\pi_s-\pi_s^*|^2\mathrm{d}s\right]^{\frac{1}{2}}<\infty.\end{aligned}$$

This finishes the proof. □

We show in Theorem 1.2.2 that if (1.2) has an optimal solution $\pi^*\in\Pi^x$, then there exists an adapted solution to the FBSDE (1.12). As a by-product we show that the optimization singles out a "pricing measure" under which the asset prices and the marginal utilities are martingales. In this sense, the process Y captures the impact of future trading gains on the agent's marginal utilities. If we assume additional conditions on the utility function U and the endowment H, we get the following regularity properties of the solution (X,Y,Z).

Proposition 1.2.1. *Let $H\in L^\infty(\Omega,\mathcal{F}_T,\mathbb{P})$ and assume that the FBSDE* (1.12) *admits an adapted solution (X,Y,Z). Let*

$$\varphi_1(x):=\frac{U'(x)}{U''(x)},\quad \varphi_2(x):=\frac{U^{(3)}(x)|U'(x)|^2}{(U''(x))^3},\quad \varphi_3(x):=\frac{U^{(3)}(x)}{U''(x)},\quad x\in\mathbb{R}.$$

Assume that U is such that φ_i, $i=1,2,3$, are bounded and Lipschitz continuous functions.

Then (X,Y,Z) *is the unique solution of* (1.12) *in* $\mathbb{S}^2(\mathbb{R})\times\mathbb{S}^\infty(\mathbb{R})\times\mathbb{H}^2(\mathbb{R}^d)$*. In addition,* $Z\cdot W$ *is a BMO-martingale.*

Proof. Let (X,Y,Z) be a solution to (1.12). Then, using Morlais [96, Theorem 2.5 and Lemma 3.1] we have for the backward equation of the FBSDE that Y is bounded, that Z is in $\mathbb{H}^2(\mathbb{R}^d)$ and that $Z\cdot W$ is a BMO-martingale. In addition there exists a unique solution to the backward component in this space for a given process X. Now the regularity properties of the processes (Y,Z) imply that X is in $\mathbb{S}^2(\mathbb{R})$. We turn to the uniqueness of the X process. Assume that there exists another solution (X',Y',Z') of (1.12). Hence, Theorem 1.2.3 implies that $\pi^{*\prime} := -\frac{U'(X'+Y')}{U''(X'+Y')}\theta^i + Z'^i, i\in\{1,\dots,d_1\}$ is an optimal solution to the problem (1.2) and X' is the optimal wealth process. However, by strict concavity of U and by convexity of Π^x the optimal strategy has to be unique. So X and X' are the wealth processes of the same optimal strategy, thus they have to coincide. This implies $(Y',Z')=(Y,Z)$. □

In the complete case we are able to explicitly construct the solution (X,Y,Z). We elaborate on this in the following section.

1.2.2 Characterization and verification: complete markets

Let us consider the case of a complete market. We assume $d=1$ for simplicity, and H denotes a square integrable random variable measurable with respect $\mathcal{F}_T$.

In the complete case we can give sufficient conditions for the existence of a solution to the system (1.12). In the following remark, we make an observation which becomes important in the construction of solutions.

Remark 1.2.3. *Using* (1.12) *the martingale* $U'(X^{\pi^*}+Y)$ *becomes more explicit because Itô's formula applied to* $U'(X^{\pi^*}+Y)$ *yields*

$$\begin{aligned} U'(X_t^{\pi^*}+Y_t) &= U'(x+Y_0)+\int_0^t U''(X_s^{\pi^*}+Y_s)(\pi_s^*+Z_s)\mathrm{d}W_s \\ &= U'(x+Y_0)-\int_0^t U'(X_s^{\pi^*}+Y_s)\theta_s\mathrm{d}W_s, \quad t\in[0,T], \end{aligned}$$

where the second line follows from the representation for π^* *from Theorem 1.2.1. Hence,*

$$U'(X_t^{\pi^*}+Y_t)=U'(x+Y_0)\mathcal{E}(-\theta\cdot W)_t, \quad t\in[0,T]. \tag{1.15}$$

Moreover, if (X,Y,Z) *is an adapted solution to the system* (1.12)*, then* $P:=X+Y$ *solves the forward SDE*

$$P_t = x+Y_0-\int_0^t\theta_s\frac{U'(P_s)}{U''(P_s)}\mathrm{d}W_s-\int_0^t\frac{1}{2}|\theta_s|^2\frac{U^{(3)}(P_s)|U'(P_s)|^2}{(U''(P_s))^3}\mathrm{d}s, \quad t\in[0,T]. \tag{1.16}$$

In addition, if $(X,Y,Z)\in\mathbb{S}^2(\mathbb{R})\times\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R}^d)$*, then* $P\in\mathbb{S}^2(\mathbb{R})$*. Thus a necessary condition for the FBSDE* (1.12) *to have a solution is that the SDE* (1.16) *admits a solution.*

This remark allows to prove the existence of a solution to the system (1.12) under a condition on the risk aversion coefficient $-\frac{U''}{U'}$. In the following, we state an existence result for the FBSDE (1.12) that characterizes optimal trading strategies in terms of the functions $\varphi_1(x) = \frac{U'(x)}{U''(x)}$ and $\varphi_2(x) = \frac{U^{(3)}(x)|U'(x)|^2}{(U''(x))^3}$.

Proposition 1.2.2. *Assume that the functions φ_1 and φ_2 are bounded and Lipschitz continuous. Let $H \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$. Then, for $t \in [0,T]$ and $x > 0$, the FBSDE*

$$\begin{cases} X_t = x - \int_0^t \left(\theta_s \frac{U'(X_s+Y_s)}{U''(X_s+Y_s)} + Z_s\right) \mathrm{d}W_s - \int_0^t \left(\theta_s \frac{U'(X_s+Y_s)}{U''(X_s+Y_s)} + Z_s\right) \cdot \theta_s \mathrm{d}s, \\ Y_t = H - \int_t^T Z_s \mathrm{d}W_s - \int_t^T \left(-\frac{1}{2}|\theta_s|^2 \frac{U^{(3)}(X_s+Y_s)|U'(X_s+Y_s)|^2}{(U''(X_s+Y_s))^3} + |\theta_s|^2 \frac{U'(X_s+Y_s)}{U''(X_s+Y_s)} + Z_s \cdot \theta_s \right) \mathrm{d}s \end{cases} \tag{1.17}$$

admits a solution (X,Y,Z) with values in $\mathbb{S}^2(\mathbb{R}) \times \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^d)$ such that $\mathbb{E}\big[|U(X_T + H)|\big] < \infty$ and $\mathbb{E}\big[|U'(X_T + H)|^2\big] < \infty$.

Proof. Let $m \in \mathbb{R}$ and consider the SDE

$$P_t^m = x + m - \int_0^t \theta_s \varphi_1(P_s^m) \mathrm{d}W_s - \int_0^t \frac{1}{2}|\theta_s|^2 \varphi_2(P_s^m) \mathrm{d}s, \quad t \in [0,T].$$

Due to the boundedness of θ, this SDE has Lipschitz continuous coefficients, thus existence and uniqueness of a solution in $\mathbb{S}^2(\mathbb{R})$ is straightforward (see e.g. Protter [114, V.3. Lemma 1]). Next, consider the BSDE

$$Y_t^m = H - \int_t^T Z_s^m \mathrm{d}W_s - \int_t^T \left(-\frac{1}{2}|\theta_s|^2 \varphi_2(P_s^m) + |\theta_s|^2 \varphi_1(P_s^m) + Z_s^m \cdot \theta_s\right) \mathrm{d}s. \tag{1.18}$$

We denote its driver by $f(s,p,z) := -\frac{1}{2}|\theta_s|^2 \varphi_2(p) + |\theta_s|^2 \varphi_1(p) + z \cdot \theta_s$. Using the boundedness of φ_1, φ_2 and θ, there exists a constant $K > 0$ such that

$$|f(s,p,z)| \le K(1 + |z|).$$

Note that this constant in particular does not depend on m. Since the driver f is Lipschitz in z, a classical result from Pardoux and Peng [102] yields a unique pair of adapted processes (Y^m, Z^m) in $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^d)$ solving (1.18). In addition, classical a priori estimates (see e.g. Ma and Zhang [88, Lemma 2.2]) yield that $|Y_t^m| \le 1 + K$ holds $\mathbb{P}$-a.s. for all $t \in [0,T]$. Recalling that K does not depend on m, we thus have $|Y_0^m| \le 1 + K$. By standard arguments one can show that the map $m \mapsto Y_0^m$ is continuous, which we repeat here in order to make the presentation self-contained. Fix $m, m' \in \mathbb{R}$ with $m \neq m'$ and put $\delta Y_t := Y_t^m - Y_t^{m'}$, $\delta Z_t := Z_t^m - Z_t^{m'}$. By (1.18) one easily sees that $(\delta Y, \delta Z)$ is solution to the Lipschitz BSDE

$$\delta Y_t = 0 - \int_t^T \delta Z_s \mathrm{d}W_s - \int_t^T \Big(\theta_s \delta Z_s + h(s)\Big) \mathrm{d}s,$$

with $h(s) := \frac{1}{2}|\theta_s|^2 (\varphi_2(P_s^m) - \varphi_2(P_s^{m'})) + |\theta_s|^2 (\varphi_1(P_s^m) - \varphi_1(P_s^{m'}))$. Again, by classical a priori estimates for Lipschitz BSDEs we get

$$|\delta Y_0|^2 \le \mathbb{E}\Big[\sup_{t\in[0,T]} |\delta Y_t|^2\Big] \le C\mathbb{E}\Big[\int_0^T |h(s)|^2 \mathrm{d}s\Big].$$

Now θ being bounded and φ_1 and φ_2 being Lipschitz continuous imply

$$\mathbb{E}\left[\int_0^T |h(s)|^2 \mathrm{d}s\right] \le C\mathbb{E}\left[\int_0^T |P_s^m - P_s^{m'}|^2 \mathrm{d}s\right] \le C\mathbb{E}\left[\sup_{t\in[0,T]} |P_t^m - P_t^{m'}|^2\right].$$

Combining these inequalities with classical estimates on SDEs with Lipschitz continuous coefficients (see e.g. Protter [114, Estimate (***) in the proof of Theorem V.7.37]) we finally obtain

$$|\delta Y_0|^2 \le C|m - m'|^2,$$

hence, passing to the limit $m' \to m$, we show the continuity of the mapping $m \mapsto Y_0^m$. This in conjunction with $m \mapsto Y_0^m$ being bounded yields that there exists an element $m^* \in \mathbb{R}$ such that $Y_0^{m^*} = m^*$. Setting

$$X_t^{m^*} := P_t^{m^*} - Y_t^{m^*}, \quad t \in [0,T],$$

it is straightforward to check that $(X^{m^*}, Y^{m^*}, Z^{m^*})$ satisfies (1.17). Moreover, we have $X^{m^*} \in \mathbb{S}^2(\mathbb{R})$ since Y^{m^*} is bounded and since $P^{m^*} \in \mathbb{S}^2(\mathbb{R})$. Now let us denote $X = X^{m^*}$. According to (1.15), we have $U'(X_T + Y_T) = U'(x+m)\mathcal{E}(-\theta \cdot W)$, thus by the BDG inequality we have

$$\mathbb{E}[|U'(X_T + Y_T)|^2] \le C\mathbb{E}\Big[\int_0^T |\theta_s|^2 \mathrm{d}s\Big] < \infty,$$

proving $\mathbb{E}[|U'(X_T + Y_T)|^2] < \infty$. Since U is concave we have

$$U(z) \le U'(0)z + U(0), \quad z \in \mathbb{R}.$$

Consequently[1], we have

$$\begin{aligned}\mathbb{E}[|U(X_T + H)|] &\le \mathbb{E}[|U'(0)|\ |X_T + H| + |U(0)|] \\ &\le C\Big(1 + \mathbb{E}\big[|H|^2\big] + \mathbb{E}\big[\sup_{t\in[0,T]} |X_t|^2\big]\Big) \\ &< \infty,\end{aligned}$$

which concludes the proof. □

[1]Since U is a concave continuous function defined on the entire real line, $U'(0)$ must exist.

1.3 Utility functions on the positive half-line

In this section we study utility functions $U : I \to \mathbb{R}$ defined on the positive half-line $I = (0, \infty)$. Again, we assume that U is strictly increasing and strictly concave.

In the previous section we have derived an FBSDE characterization of the optimal strategy for the utility maximization problem (1.2). The key observation there is that there exists a stochastic process Y such that $U'(X^{\pi^*} + Y)$ is a martingale. However, if U is only defined on the positive half-line, it is not clear that the quantity $U'(X^{\pi^*} + Y)$ gives rise to something meaningful. We could generalize this approach by looking for a function Φ such that $\Phi(X^{\pi^*}, Y)$ is a martingale and such that $\Phi(X^{\pi^*}_T, Y_T) = U'(X^{\pi^*}_T + H)$. When $H = 0$, it turns out that a good choice of function Φ is $\Phi(x, y) := U'(x)\exp(y)$ since the system we obtain coincides (up to a non-linear transformation) with the one obtained in Peng [107, Section 4] by means of the stochastic maximum principle. We remark that the system of Peng [107] is not formulated as an FBSDE but rather as a system involving the wealth process whose dynamics depends on the strategy and an adjoint equation stemming from a Hamiltonian. However, a reformulation of these equations allows to obtain an FBSDE. We give more details about this approach in Section 1.4.1.

In the previous section, π denotes the total amount of money invested into the stock, i.e. the number of shares are given by $\pi/\tilde{S}$. Now we denote by π^i the proportion of wealth invested in the i-th stock S^i. Again we denote by Π^x the set of admissible strategies with initial capital $x > 0$ which is now defined by

$$\Pi^x := \left\{ \pi : \Omega \times [0, T] \to \mathbb{R}^{d_1} \text{ predictable: } \mathbb{E}\Big[\int_0^T |\pi_s|^2 \mathrm{d}s \Big] < \infty \right\}. \tag{1.19}$$

The associated wealth process is now given by

$$X^\pi_t := x + \int_0^t X^\pi_s \pi_s \mathrm{d}S^{\mathcal{H}}_s, \quad t \in [0, T].$$

Again, we extend π to $\mathbb{R}^d$ via $\tilde{\pi} := (\pi^1, \dots, \pi^{d_1}, 0, \dots, 0)$ and make the convention that we write π instead of $\tilde{\pi}$. This gives rise to

$$X^\pi_t = x\mathcal{E}\left(\int_0^\cdot \pi_r \mathrm{d}S^{\mathcal{H}}_r \right)_t, \quad t \in [0, T].$$

From now on we consider a positive $\mathcal{F}_T$-measurable random variable H. Moreover, we need the following assumptions:

(H3) $U : \mathbb{R}_+ \to \mathbb{R}$ is three times continuously differentiable, strictly increasing and concave.

(H4) We say that assumption (H4) holds for an element π^* in Π^x, if

(i) $\mathbb{E}\Big[|X^{\pi^*}_T U'(X^{\pi^*}_T + H)|^2 \Big] < \infty$;

(ii) given some $\rho \in \Pi^x$, the family of random variables

$$\left(\frac{1}{\varepsilon}(X_T^{\pi^*+\varepsilon\rho} - X_T^{\pi^*}) \int_0^1 U'(X_T^{\pi^*} + H + r(X_T^{\pi^*+\varepsilon\rho} - X_T^{\pi^*}))\mathrm{d}r\right)_{\varepsilon\in(0,1)}$$

is uniformly integrable;

(iii) given some $\rho \in \Pi^x$,

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \mathbb{E}\left[\left|\frac{1}{\varepsilon}(X_t^{\pi^*+\varepsilon\rho} - X_t^{\pi^*})\right|^2\right] < \infty.$$

(H5) There exists a constant $c > 0$ such that $\frac{-U'(x)}{xU''(x)} \le c$ for all $x \in \mathbb{R}_+$.

Note that it follows from **(H4)**(iii) that we have

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \mathbb{E}\left[\left|\frac{1}{\varepsilon}(X_t^{\pi^*+\varepsilon\rho} - X_t^{\pi^*}) - \xi_t\right|^2\right] = 0,$$

where $\mathrm{d}\xi_t = \pi_t^*\xi_t \mathrm{d}S_t^{\mathcal{H}} + \rho_t X_t^{\pi^*}\mathrm{d}S_t^{\mathcal{H}}, t \in [0,T]$. Moreover, in condition **(H4)**, if $H \ge a > 0$ for some constant $a > 0$ is satisfied, then (iii) implies (ii).

1.3.1 Characterization and verification: incomplete markets

Similar to Section 1.2.1, we characterize the optimal strategy in terms of a pair of adapted processes (Y, Z). In the framework of this section, we have the following characterization result.

Theorem 1.3.1. *Assume that **(H3)** holds. Let H be a non-negative random variable in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ which is bounded away from zero. Let π^* be an optimal solution to* (1.2) *satisfying $\mathbb{E}\left[|U(X_T^{\pi^*} + H)|\right] < \infty$ and condition **(H4)**. Then there exists a predictable process Y with $Y_T = \log\left(U'(X_T^{\pi^*} + H)\right) - \log\left(U'(X_T^{\pi^*})\right)$ such that $X^{\pi^*}U'(X^{\pi^*})\exp(Y)$ is a martingale and the optimal strategy has the representation*

$$\pi_s^{*^i} = -\frac{U'(X_s^{\pi^*})}{X_s^{\pi^*}U''(X_s^{\pi^*})}(Z_s^i + \theta_s^i), \quad s \in [0,T], \quad i = 1, \dots, d_1,$$

where $Z_t := \left(\frac{\mathrm{d}\langle Y, W^1\rangle_t}{\mathrm{d}t}, \dots, \frac{\mathrm{d}\langle Y, W^d\rangle_t}{\mathrm{d}t}\right)$.

Proof. In analogy to the proof of Theorem 1.2.1, we show the existence of a process Y such that $X^{\pi^*}U'(X^{\pi^*})\exp(Y)$ is a martingale with terminal value $Y_T = \log(U'(X_T^{\pi^*} + H)) - \log(U'(X_T^{\pi^*}))$. As a consequence, we obtain $U'(X_T^{\pi^*} + H) = U'(X_T^{\pi^*})\exp(Y_T)$. By **(H4)**, the process

$$\alpha_t := \mathbb{E}\left[X_T^{\pi^*}U'(X_T^{\pi^*} + H)|\mathcal{F}_t\right], \quad t \in [0,T],$$

is a square integrable martingale. In addition, it is the unique solution to the BSDE

$$\alpha_t = X_T^{\pi^*} U'(X_T^{\pi^*} + H) - \int_t^T \beta_s \mathrm{d}W_s, \quad t \in [0,T],$$

where β is a square integrable predictable process with values in $\mathbb{R}^d$. We set $Y :=\log(\alpha) - \log(U'(X^{\pi^*})) - \log(X^{\pi^*})$. Itô's formula implies

$$\begin{aligned}
Y_t = Y_T &- \int_t^T \left[\frac{\beta_s}{\alpha_s} - \frac{U''(X_s^{\pi^*})}{U'(X_s^{\pi^*})} X_s^{\pi^*} \pi_s^* - \pi_s^* \right] \mathrm{d}W_s \\
&- \int_t^T \Bigg[-\frac{1}{2} \frac{|\beta_s|^2}{|\alpha_s|^2} - \left(\frac{U''(X_s^{\pi^*})}{U'(X_s^{\pi^*})} X_s^{\pi^*} \pi_s^* + \pi_s^* \right) \cdot \theta_s^{\mathcal{H}} \\
&\quad + \frac{|X_s^{\pi^*} \pi_s^*|^2}{2} \left(\left| \frac{U''(X_s^{\pi^*})}{U'(X_s^{\pi^*})} \right|^2 - \frac{U^{(3)}(X_s^{\pi^*})}{U'(X_s^{\pi^*})} \right) + \frac{|\pi_s^*|^2}{2} \Bigg] \mathrm{d}s.
\end{aligned}$$

Setting

$$Z_t^i = \frac{\beta_t^i}{\alpha_t} - \frac{\pi_t^*}{U'(X_t^{\pi^*})} (X_t^{\pi^*} U''(X_t^{\pi^*}) + U'(X_t^{\pi^*})), \quad i = 1, \dots, d, \tag{1.20}$$

we get

$$\begin{aligned}
Y_t = Y_T - \int_t^T Z_s \mathrm{d}W_s - \int_t^T \Bigg[&-\frac{1}{2} \frac{U^{(3)}(X_s^{\pi^*})}{U'(X_s^{\pi^*})} |X_s^{\pi^*} \pi_s^*|^2 \\
&- (Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}) \cdot \left(\frac{U''(X_s^{\pi^*})}{U'(X_s^{\pi^*})} X_s^{\pi^*} \pi_s^* + \pi_s^* \right) \\
&- \frac{U''(X_s^{\pi^*})}{U'(X_s^{\pi^*})} X_s^{\pi^*} |\pi_s^*|^2 - \frac{1}{2} |Z_s|^2 \Bigg] \mathrm{d}s, \quad t \in [0,T].
\end{aligned}$$

We now derive the characterization of π^* in terms of U', Y and Z. We employ a variational argument similar to Peng [107], however we replace in our context the maximum principle by a BSDE. Let us fix $\pi \in \Pi^x$. Since Π^x is a convex set, for $\rho := \pi - \pi^*$, $\pi^* + \varepsilon\rho$ remains an admissible strategy for every $\varepsilon \in (0,1)$. We have

$$\begin{aligned}
&\frac{1}{\varepsilon} (U(X_T^{\pi^* + \varepsilon\rho} + H) - U(X_T^{\pi^*} + H)) = \\
&\qquad \frac{1}{\varepsilon} (X_T^{\pi^* + \varepsilon\rho} - X_T^{\pi^*}) \int_0^1 U'(X_T^{\pi^*} + H + r(X_T^{\pi^* + \varepsilon\rho} - X_T^{\pi^*})) \mathrm{d}r.
\end{aligned}$$

Since π^* is optimal we find

$$\mathbb{E}\left[\frac{1}{\varepsilon} (X_T^{\pi^* + \varepsilon\rho} - X_T^{\pi^*}) \int_0^1 U'(X_T^{\pi^*} + H + r(X_T^{\pi^* + \varepsilon\rho} - X_T^{\pi^*})) \mathrm{d}r \right] \le 0, \quad \varepsilon > 0. \tag{1.21}$$

Now let ξ be defined as

$$\mathrm{d}\xi_t = (\pi^*_t \xi_t + \rho_t X^{\pi^*}_t)\mathrm{d}S^{\mathcal{H}}_t, \quad t \in [0,T].$$

By **(H4)**, we can apply Lebesgue's dominated convergence theorem to inequality (1.21) which (by passing to a subsequence if necessary) yields

$$\mathbb{E}[\xi_T U'(X^{\pi^*}_T + H)] = \lim_{\varepsilon \to 0} \mathbb{E}\left[\frac{1}{\varepsilon}(X^{\pi^*+\varepsilon\rho}_T - X^{\pi^*}_T) \int_0^1 U'(X^{\pi^*}_T + H + r(X^{\pi^*+\varepsilon\rho}_T - X^{\pi^*}_T))\mathrm{d}r\right].$$

In combination with (1.21), this gives rise to

$$\mathbb{E}[\xi_T (X^{\pi^*}_T)^{-1} U'(X^{\pi^*}_T) X^{\pi^*}_T \exp(Y_T)] = \mathbb{E}[\xi_T U'(X^{\pi^*}_T + H)] \le 0, \quad \pi \in \Pi^x. \tag{1.22}$$

We now restrict our attention to a particular class of processes π, that is, we choose ρ to be a bounded predictable process and we define $\pi := \rho + \pi^*$ which is an admissible strategy since it is square integrable. The integration by parts formual yields

$$\xi_t (X^{\pi^*}_t)^{-1} = \int_0^t \rho_s \mathrm{d}W^{\mathcal{H}}_s + \int_0^t [\rho_s \cdot \theta^{\mathcal{H}}_s - \rho_s \cdot \pi^*_s]\mathrm{d}s, \quad t \in [0,T].$$

Another application of integration by parts to $\alpha = U'(X^{\pi^*})X^{\pi^*}\exp(Y)$ and $\xi(X^{\pi^*})^{-1}$ yields

$$\begin{aligned}
\xi_T U'(X^{\pi^*}_T + Y_T) &= \xi_T (X^{\pi^*}_T)^{-1} U'(X^{\pi^*}_T) X^{\pi^*}_T \exp(Y_T) \\
&= \int_0^T \xi_t (X^{\pi^*}_t)^{-1} \mathrm{d}\alpha_t + \int_0^T \alpha_t \rho_t \mathrm{d}W^{\mathcal{H}}_t \\
&\quad + \int_0^T \rho_t \exp(Y_t) X^{\pi^*}_t \cdot (U'(X^{\pi^*}_t)(Z^{\mathcal{H}}_t + \theta^{\mathcal{H}}_t) + U''(X^{\pi^*}_t) X^{\pi^*}_t \pi^*_t)\mathrm{d}t.
\end{aligned} \tag{1.23}$$

In order to take expectations in the above relation, we prove some auxiliary moment estimates. Using that ρ is bounded, we get

$$\begin{aligned}
\mathbb{E}\Big[\sup_{t\in[0,T]} |\xi_t (X^{\pi^*}_t)^{-1}|^2\Big] &= \mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \rho_s \mathrm{d}W^{\mathcal{H}}_s + \int_0^t (\rho_s \cdot \theta^{\mathcal{H}}_s - \rho_s \cdot \pi^*_s)\mathrm{d}s\right|^2\right] \\
&\le C\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \rho_s \mathrm{d}W^{\mathcal{H}}_s\right|^2\right] + \mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t |\rho_s \cdot \theta^{\mathcal{H}}_s - \rho_s \cdot \pi^*_s|\mathrm{d}s\right|^2\right] \\
&\le C\left(\mathbb{E}\left[\int_0^T |\rho_s|^2 \mathrm{d}s\right] + \mathbb{E}\left[\left|\int_0^T \rho_s \cdot \theta^{\mathcal{H}}_s \mathrm{d}s\right|^2\right] + \mathbb{E}\left[\left|\int_0^T \rho_s \cdot \pi^*_s \mathrm{d}s\right|^2\right]\right) \\
&\le C\left(1 + \mathbb{E}\left[\int_0^T |\pi^*_s|^2 \mathrm{d}s\right]\right) < \infty,
\end{aligned} \tag{1.24}$$

where we used Doob's inequality. Consequently, we obtain

$$\mathbb{E}\Big[|\xi_T(X_T^{\pi^*})^{-1}\alpha_T|\Big] \le \mathbb{E}\Big[|\alpha_T|^2\Big]^{1/2}\mathbb{E}\Big[|\xi_T(X_T^{\pi^*})^{-1}|^2\Big]^{1/2} < \infty,$$

which follows from the Cauchy-Schwarz inequality. With ρ being bounded, it follows that

$$\mathbb{E}\left[\int_0^T |\alpha_s\rho_s|^2 \mathrm{d}s\right] \le C\mathbb{E}\left[\int_0^T |\alpha_s|^2 \mathrm{d}s\right] < \infty.$$

Hence $\int_0^\cdot \alpha_t\rho_t \mathrm{d}W_t^{\mathcal{H}}$ is a square integrable martingale. Next, let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence for the local martingale $\int_0^\cdot \xi_t(X_t^{\pi^*})^{-1}\mathrm{d}\alpha_t$. We clearly have

$$\left|\int_0^{\tau_n} \xi_t(X_t^{\pi^*})^{-1}\mathrm{d}\alpha_t\right| \le \sup_{t\in[0,T]}\left|\int_0^t \xi_s(X_s^{\pi^*})^{-1}\mathrm{d}\alpha_s\right|.$$

To apply Lebesgue's dominated convergence theorem and to show that

$$\mathbb{E}\left[\int_0^T \xi_t(X_t^{\pi^*})^{-1}\mathrm{d}\alpha_t\right] = 0,$$

we need to prove $\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \xi_s(X_s^{\pi^*})^{-1}\mathrm{d}\alpha_s\right|\right] < \infty$. We see because we have

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \xi_t(X_t^{\pi^*})^{-1}\mathrm{d}\alpha_t\right|\right] &\le C\mathbb{E}\left[\left|\int_0^T |\xi_t|^2|(X_t^{\pi^*})^{-1}|^2\mathrm{d}\langle\alpha\rangle_t\right|^{1/2}\right]\\ &\le C\mathbb{E}\left[\sup_{t\in[0,T]}|\xi_t|^2|(X_t^{\pi^*})^{-1}|^2\right]^{1/2}\mathbb{E}\left[\langle\alpha\rangle_T\right]^{1/2}\\ &< \infty,\end{aligned}$$

where we have used the estimate (1.24). Thus, by (1.23) it follows that

$$\mathbb{E}\left[\left|\int_0^T \rho_t\exp(Y_t)X_t^{\pi^*}\cdot(U'(X_t^{\pi^*})(Z_t^{\mathcal{H}}+\theta_t^{\mathcal{H}})+U''(X_t^{\pi^*})X_t^{\pi^*}\pi_t^*)\mathrm{d}t\right|\right] < \infty,$$

and from (1.22), it holds that for every π in Π^x such that ρ is bounded, we get

$$\mathbb{E}\left[\int_0^T \rho_t\exp(Y_t)X_t^{\pi^*}\cdot(U'(X_t^{\pi^*})(Z_t^{\mathcal{H}}+\theta_t^{\mathcal{H}})+U''(X_t^{\pi^*})X_t^{\pi^*}\pi_t^*)\mathrm{d}t\right] \le 0.$$

Substituting ρ with $-\rho$ in the previous inequality, we obtain for every ρ

$$\mathbb{E}\left[\int_0^T \rho_t\exp(Y_t)X_t^{\pi^*}\cdot(U'(X_t^{\pi^*})(Z_t^{\mathcal{H}}+\theta_t^{\mathcal{H}})+U''(X_t^{\pi^*})X_t^{\pi^*}\pi_t^*)\mathrm{d}t\right] = 0. \tag{1.25}$$

Now let $A_t := U'(X_t^{\pi^*})(Z_t^{\mathcal{H}}+\theta_t^{\mathcal{H}})+U''(X_t^{\pi^*})X_t^{\pi^*}\pi_t^*$ and let $\rho_t := \mathbb{1}_{A_t>0}$ for $t\in[0,T]$.

Recall that $\mathrm{d}\mathbb{P}\otimes\mathrm{d}t$-a.s. we have $\exp(Y)X^{\pi^*}>0$. Plugging ρ into (1.25) yields

$$A_t \le 0,\ \mathrm{d}\mathbb{P}\otimes\mathrm{d}t - a.s.$$

Similarly, choosing $\rho_t := \mathbb{1}_{A_t<0}$, we find

$$A_t = 0,\ \mathrm{d}\mathbb{P}\otimes\mathrm{d}t - a.s.$$

Thus, we have achieved

$$\pi_t^{*^i} = -\frac{U'(X_t^{\pi^*})}{X_t^{\pi^*}U''(X_t^{\pi^*})}(Z_t^i+\theta_t^i),\quad t\in[0,T],\quad i=1,\dots,d_1,$$

which concludes the proof ☐

We can also formulate the converse implication.

Theorem 1.3.2. *Assume **(H3)** and **(H5)**. Let (X,Y,Z) be an adapted solution of the FBSDE*

$$\begin{cases} X_t = x - \int_0^t \frac{U'(X_s)}{U''(X_s)}(Z_s^{\mathcal{H}}+\theta_s^{\mathcal{H}})\mathrm{d}W_s^{\mathcal{H}} - \int_0^t \frac{U'(X_s)}{U''(X_s)}(Z_s^{\mathcal{H}}+\theta_s^{\mathcal{H}})\theta_s\mathrm{d}s, \quad x>0, \\ Y_t = \log\left(\frac{U'(X_T+H)}{U'(X_T)}\right) - \int_t^T \left[(|Z_s^{\mathcal{H}}+\theta_s^{\mathcal{H}}|^2)\left(1-\frac{1}{2}\frac{U^{(3)}(X_s)U'(X_s)}{|U''(X_s)|^2}\right) - \frac{1}{2}|Z_s|^2\right]\mathrm{d}s \\ \qquad\qquad - \int_t^T Z_s\mathrm{d}W_s, \end{cases} \tag{1.26}$$

such that we have $\mathbb{E}[|U(X_T+H)|]<\infty$, $Z\in\mathbb{H}^2(\mathbb{R}^d)$ and the positive local martingale $XU'(X)\exp(Y)$ is a true martingale. Then,

$$\pi_s^{*^i} := -\frac{U'(X_s)}{X_sU''(X_s)}(Z_s^i+\theta_s^i),\quad s\in[0,T],\quad i=1,\dots,d_1$$

is a solution to the optimization problem (1.2).

Proof. We first note that $\pi^*\in\Pi^x$ since by the fact that $Z\in\mathbb{H}^2(\mathbb{R}^d)$, there is a constant $C>0$ such that

$$\mathbb{E}\left[\int_0^T|\pi_t^*|^2\mathrm{d}t\right]\le C\ \mathbb{E}\left[\int_0^T|Z_t^{\mathcal{H}}+\theta_t^{\mathcal{H}}|^2\mathrm{d}t\right]<\infty.$$

Now let π be an element of Π^x. Let $D:=U'(X)\exp(Y)$. Applying Itô's formula and plugging in the expression for π^*, we find that

$$\mathrm{d}D_t = D_t(-\theta_t\mathrm{d}W_t^{\mathcal{H}} + Z_t\mathrm{d}W_t^{\mathcal{O}}),\quad D_0 = U'(x)\exp(Y_0).$$

Hence,

$$D_t = U'(x)\exp(Y_0)\mathcal{E}\left(-\int_0^{\cdot}\theta_s\mathrm{d}W_s^{\mathcal{H}} + \int_0^{\cdot}Z_s\mathrm{d}W_s^{\mathcal{O}}\right)_t,\quad t\in[0,T], \tag{1.27}$$

is a non-negative local martingale. Now fix π in Π^x. The product formula implies for $X^\pi D$ that

$$X^\pi D = x D_0 \mathcal{E}\Big((\pi - \theta)\cdot W^{\mathcal{H}} + Z \cdot W^{\mathcal{O}}\Big),$$

hence, $X^\pi D$ is a supermartingale implying $\mathbb{E}[X^\pi_T D_T] \le D_0 x$. By hypothesis, $XD = XU'(X)\exp(Y)$ is a true martingale, thus $\mathbb{E}[X_T D_T] = D_0 x$. Combining these facts, recalling that $D_T = U'(X_T + H)$ and using the concavity of U, we obtain

$$\mathbb{E}[U(X^\pi_T + H) - U(X_T + H)] \le \mathbb{E}[U'(X_T + H)(X^\pi_T - X_T)] \le 0. \tag{1.28}$$

This finishes the proof. □

Remark 1.3.1. *Using the notation $X = X^{\pi^*}$, we see in the previous proof that the quantity $U'(X)\exp(Y)(X^\pi - X^{\pi^*})$ satisfies*

$$U'(X^\pi)\exp(Y)(X^\pi - X^{\pi^*}) = \int_0^\cdot (X^\pi_t - X^{\pi^*}_t)\mathrm{d}D_t + \int_0^\cdot D_t(\pi_t X^\pi_t - \pi^*_t X^{\pi^*}_t)\mathrm{d}W^{\mathcal{H}}_t,$$

thus $U'(X^\pi)\exp(Y)(X^\pi - X^{\pi^*})$ is a local martingale for every admissible strategy π.

Remark 1.3.2. *Note that using the regularity assumptions of the FBSDE* (1.26)*, we have derived that $D := U'(X^{\pi^*})\exp(Y)$ is the true martingale*

$$D_t = U'(x)\exp(Y_0)\mathcal{E}\left(-\theta^{\mathcal{H}}\cdot W^{\mathcal{H}} + Z^{\mathcal{O}}\cdot W^{\mathcal{O}}\right), \quad t \in [0,T].$$

1.3.2 Characterization and verification: complete markets

We adopt the simplifications from Section 1.3 and consider the case $d = 1$ and $H = 0$. In the complete case we can give sufficient conditions for the existence of a solution to the system (1.26). We give the following remark which resembles Remark 1.3.2.

Remark 1.3.3. *An application of Itô's formula to $U'(X^{\pi^*})\exp(Y)$ gives rise to*

$$U'(X^{\pi^*}_t)\exp(Y_t) = U'(x)\exp(Y_0) - \int_0^t U'(X_s)\exp(Y_s)\theta_s \mathrm{d}W_s,$$

hence, we have

$$U'(X^{\pi^*}_t)\exp(Y_t) = U'(x)\exp(Y_0)\mathcal{E}(-\theta\cdot W)_t, \quad t\in[0,T]. \tag{1.29}$$

This observation allows to prove the existence of a solution of (1.26) under a condition on the risk aversion coefficient $-\frac{U''}{U'}$. Let $\varphi_1(x) := \frac{U'(x)}{U''(x)}$ and $\varphi_2(x) := 1 - \frac{1}{2}\frac{U^{(3)}(x)U'(x)}{|U''(x)|^2}$. We state the following remark.

Remark 1.3.4. *Note that if φ_2 is constant then the system above decouples. An elementary analysis shows that this happens if and only is U is the exponential, power, logarithmic or quadratic (mean-variance hedging) function. If $U(x) = -\exp(-\alpha_1 x) - \exp(-\alpha_2 x)$ then φ_2 is bounded and Lipschitz, however this function define on the entire real line. If $U(x) := \frac{x^{\gamma_1}}{\gamma_1} + \frac{x^{\gamma_2}}{\gamma_2}, x > 0$, then φ_2 is a bounded function.*

We now give a sufficient condition for the system (1.26) to exhibit a solution.

Theorem 1.3.3. *Assume that φ_2 is a continuous and bounded function. Then there exists an adapted solution $(X, Y, Z) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$ to the FBSDE*

$$\begin{cases} X_t = x - \int_0^t \frac{U'(X_s)}{U''(X_s)}(Z_s + \theta_s)\mathrm{d}W_s - \int_0^t \frac{U'(X_s)}{U''(X_s)}(Z_s + \theta_s)\theta_s \mathrm{d}s, \quad x > 0, \\ Y_t = 0 - \int_t^T Z_s \mathrm{d}W_s - \int_t^T \left[|Z_s + \theta_s|^2 \left(1 - \frac{1}{2}\frac{U^{(3)}(X_s)U'(X_s)}{|U''(X_s)|^2}\right) - \frac{1}{2}|Z_s|^2\right]\mathrm{d}s. \end{cases} \tag{1.30}$$

Moreover, we have $\mathbb{E}\big[U(X_T)\big] < \infty$ and $\mathbb{E}\big[U'(X_T)^2\big] < \infty$.

Proof. Fix $m > 0$ and consider the BSDE

$$\begin{aligned} Y_t^m = 0 - \int_t^T \Bigg[& |Z_s^m + \theta_s|^2 \varphi_2\Big((U')^{-1}\big(U'(x)\exp(m)\mathcal{E}(-\theta\cdot W)_t \exp(-Y_t^m)\big)\Big) \\ & - \frac{1}{2}|Z_s^m|^2 \mathrm{d}s\Bigg] - \int_t^T Z_s^m \mathrm{d}W_s. \end{aligned}$$

Since φ_2 is bounded, the driver of the BSDE above in (Y^m, Z^m) can be bounded uniformly in m, hence Kobylanski [77] yields a pair $(Y^m, Z^m) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$ which solves the equation. Moreover, Y^m is bounded by constant $C > 0$ which does not depend on m and $Z \cdot W$ is a BMO-martingale. Mimicking the arguments from the proof of Proposition 1.2.2 we get that $m \mapsto Y_0^m$ is a continuous map. Thus there exists an element $m^* > 0$ such that $Y_0^{m^*} = m^*$. Now applying Itô's formula to

$$X^{m^*} := (U')^{-1}(U'(x)\exp(m^*)\mathcal{E}(-\theta\cdot W)\exp(-Y^m)),$$

we check that $(X^{m^*}, Y^{m^*}, Z^{m^*})$ satisfies (1.30).
Since θ is bounded, $W^\theta := W + \int_0^\cdot \theta_s \mathrm{d}s$ is a Brownian motion under the equivalent probability measure $\frac{\mathrm{d}\mathbb{P}^\theta}{\mathrm{d}\mathbb{P}} := \mathcal{E}\big(-\theta * W\big)_T$. Under $\mathbb{P}^\theta$, X becomes a true martingale,

$$X_t = x\mathcal{E}\Big(-\frac{U'(X)}{U''(X)X}(Z + \theta) * W^\theta\Big)_t, \quad t \in [0, T].$$

Since U is concave, $U(X)$ is a $\mathbb{P}^\theta$-supermartingale, hence

$$\mathbb{E}^\theta\big[U(X_T)\big] \leq U(x) < \infty.$$

However, since $\mathbb{P}^\theta$ is equivalent to $\mathbb{P}$, it follows that

$$\mathbb{E}\big[U(X_T)\big] < \infty.$$

To check $\mathbb{E}\big[U'(X_T)^2\big] < \infty$, note that $U'(X_T) = U'(X_T)e^{Y_T}$ holds a.s. According to

(1.29), we have

$$\mathbb{E}\big[U'(X_T)^2\big] = \mathbb{E}\big[U'(X_T)^2 e^{2Y_T}\big] \le C\mathbb{E}\big[\mathcal{E}(-\theta * W)^2\big] \le C\mathbb{E}\Big[\int_0^T |\theta_s|^2 \mathrm{d}s\Big] < \infty,$$

which finishes the proof. □

1.4 Links to other approaches

In this section we compare our approach of characterizing optimal investment strategies to those based on the stochastic maximum principle and on the convex duality approach.

1.4.1 Stochastic maximum principle

We depict the link of our approach in the complete market setting to an approach which employs the stochastic maximum principle. As this section is solely of an illustrative character, we only give a formal derivation. In particular, we assume here that U and U^{-1} are sufficiently smooth functions with bounded and continuous derivatives. Moreover, we confine the consideration to the complete market case with $d_1 = d = 1$ and $H = 0$. The wealth process is given by

$$X_t^\pi = x + \int_0^t \pi_s \mathrm{d}W_s + \int_0^t \pi_s\theta_s \mathrm{d}s, \quad t \in [0,T].$$

We consider $J(\pi) := \mathbb{E}[U(X_T^\pi)]$ and set $\tilde{X}^\pi := U(X^\pi)$ for which Itô's formula yields

$$\mathrm{d}\tilde{X}_t^\pi = U'(U^{-1}(\tilde{X}_t^\pi))\pi_t \mathrm{d}W_t + \Big[U'(U^{-1}(\tilde{X}_t^\pi))\pi_t\theta_t + \frac{1}{2}U''(U^{-1}(\tilde{X}_t^\pi))|\pi_t|^2\Big]\mathrm{d}t, \quad t \in [0,T],$$

and $J(\pi) = \mathbb{E}[\tilde{X}_T^\pi]$. Applying the maximum principle from Peng [107, Section 4], we get the system of controlled diffusions $\tilde{X}^\pi$ and its adjoint equation p,

$$\begin{aligned}
\mathrm{d}\tilde{X}_t^\pi &= U'(U^{-1}(\tilde{X}_t^\pi))\pi_t \mathrm{d}W_t + \Big[U'(U^{-1}(\tilde{X}_t^\pi))\pi_t\theta_t + \frac{1}{2}U''(U^{-1}(\tilde{X}_t^\pi))|\pi_t|^2\Big]\mathrm{d}t,\\
\mathrm{d}p_t &= -\Big[\Big(\frac{U''}{U'}(U^{-1}(\tilde{X}_t^\pi))\theta_t\pi_t + \frac{1}{2}\frac{U^{(3)}}{U''}(U^{-1}(\tilde{X}_t^\pi))|\pi_t|^2\Big)p_t + k_t\frac{U''}{U'}(U^{-1}(\tilde{X}_t^\pi))\pi_t\Big]\mathrm{d}t\\
&\quad + k_t \mathrm{d}W_t,
\end{aligned} \tag{1.31}$$

where $\tilde{X}_0^\pi = U(x)$ and $p_T = 1$. The corresponding Hamiltonian is given by

$$H(t,x,\pi,p,k) := p[U'(U^{-1}(x))\pi\theta_t + \frac{1}{2}U''(U^{-1}(x)|\pi|^2] + kU'(U^{-1}(x))\pi.$$

Formally maximizing $H(t,x,\pi,p,k)$ yields as a candidate for the optimal strategy π^*

$$\pi_t^* := -\frac{U'}{U''}(U^{-1}(\tilde{X}_t^\pi))\left[\frac{k_t}{p_t} + \theta_t\right].$$

Plugging this into (1.31) yields

$$\begin{cases} \mathrm{d}\tilde{X}_t^{\pi^*} = -\frac{|U'|^2}{U''}(U^{-1}(\tilde{X}_t^{\pi^*}))\left(\frac{k_t}{p_t}+\theta_t\right)\left[\mathrm{d}W_t - \frac{1}{2}\left(\frac{k_t}{p_t}-\theta_t\right)\mathrm{d}t\right],\ \tilde{X}_0^{\pi} = U(x), \\ \mathrm{d}p_t = -\left(\frac{k_t}{p_t}+\theta_t\right)^2 p_t\left[-1+\frac{1}{2}\frac{U^{(3)}U'}{|U''|^2}(U^{-1}(\tilde{X}_t^{\pi}))\right]\mathrm{d}t + k_t\mathrm{d}W_t,\ p_T = 1, \end{cases} \tag{1.32}$$

for $t \in [0,T]$. We now relate this system with (1.30) using an exponential transformation. Note that the above system rewrites in the forward component as

$$\begin{cases} \mathrm{d}X_t^{\pi^*} = -\frac{U'}{U''}(X_t^{\pi^*})\left[\frac{k_t}{p_t}+\theta_t\right](\mathrm{d}W_t+\theta_t\mathrm{d}t),\ X_0^{\pi^*} = x, \\ \mathrm{d}p_t = -\left(\frac{k_t}{p_t}+\theta_t\right)^2 p_t\left[-1+\frac{1}{2}\frac{U^{(3)}U'}{|U''|^2}(X_t^{\pi^*})\right]\mathrm{d}t + k_t\mathrm{d}W_t,\ p_T = 1. \end{cases} \tag{1.33}$$

Next consider the system

$$\begin{cases} \mathrm{d}X_t^{\pi^*} = -\frac{U'}{U''}(X_t^{\pi^*})(Z_t+\theta_t)(\mathrm{d}W_t+\theta\mathrm{d}t),\ X_0^{\pi^*} = x, \\ \mathrm{d}Y_t = \left[(Z_t+\theta_t)^2\left(1-\frac{1}{2}\frac{U^{(3)}(X_t^{\pi^*})U'(X_t^{\pi^*})}{|U''|^2(X_t^{\pi^*})}\right) - \frac{1}{2}|Z_t|^2\right]\mathrm{d}t + Z_t\mathrm{d}W_t,\ Y_T = 0. \end{cases} \tag{1.34}$$

The exponential transformation now consists of setting $\tilde{p} := \exp(Y)$, $\tilde{k} := Z\tilde{p}$ and $\tilde{X} := X$. Then, Itô's formula implies that $(\tilde{p},\tilde{k})$ is a solution to (1.33).

1.4.2 BSDE solution via convex duality methods

Let us now turn to the link to the approach based on convex dual techniques for utility functions defined on the positive half-line. We have seen in Section 1.2 and Section 1.3 that our approach relies on choosing a process Y such that the quantities $U'(X^{\pi^*}+Y)$, resp. $X^{\pi^*}U'(X^{\pi^*})\exp(Y)$ are martingales. In fact, these martingales are not any martingales. For instance, in case of a utility function on the whole real line, $U'(X^{\pi^*}+Y)$ is identical to $U'(x+Y_0)\mathcal{E}(-\theta\cdot W^{\mathcal{H}}+\frac{U''}{U'}(X^{\pi^*}+Y)Z^{\mathcal{O}}\cdot W^{\mathcal{O}})$. So in the complete case it is exactly the martingale under which the price process itself is a martingale. For utility functions defined on the positive half-line this leads to the formulation of a dual problem. It is known from e.g. Karatzas et al. [73] and Kramkov and Schachermayer [80] that (under some growth conditions on U) the optimal wealth process X^{π^*} and the stochastic process Y^* that solve the dual problem are such that $X^{\pi^*}Y^*$ is a martingale. In addition, in our notation, it is a structural property of the dual approach that Y^* is of the form $Y^* = Y_0^*\mathcal{E}(-\theta^{\mathcal{H}}\cdot W^{\mathcal{H}}+M)$ where M is a martingale orthogonal to $W^{\mathcal{H}}$. Recall that in our case $X^{\pi^*}U'(X^{\pi^*})\exp(Y)$ is a martingale and in (1.27), we have proved that $D := U'(X^{\pi^*})\exp(Y)$ is of the form $D_0\mathcal{E}(-\theta^{\mathcal{H}}\cdot W^{\mathcal{H}}+Z^{\mathcal{O}}\cdot W^{\mathcal{O}})$. In other words, we have $Y^* = D$ and the $Z^{\mathcal{O}}$ component appearing in the solution of our FBSDE represents the orthogonal part of the dual optimizer in the language of the convex dual approach.

The aim of this section is to derive a solution of the forward-backward equation (1.30) by means of the results from the convex dual approach to (1.2). To this end, we recall the dual problem. Denote by Π^1 the set of admissible strategies with initial capital given by one unit of currency. The solution to the concave optimization problem (1.2) is achieved by formulating and solving the following dual problem. Denote the convex conjugate of the concave function U by

$$V(y) := \sup_{x>0} \{U(x) - xy\}, \quad y > 0,$$

and consider wealth processes $\mathrm{d}X_t^\pi = \pi_t X_t^\pi \frac{\mathrm{d}\tilde{S}_t}{\tilde{S}_t}$, $X_0^\pi = x > 0$. Next, consider the family of non-negative semimartingales

$$\mathcal{Y} := \{Y \geq 0 : Y_0 = 1, \; X^\pi Y \text{ is a supermartingale for every } \pi \in \Pi^1\}.$$

Then, the primal problem (1.2) is solved by finding a solution to the convex dual optimization problem of the type

$$v(y) = \inf_{Y_T \in \mathcal{Y}} \mathbb{E}\big[V(yY_T) + yY_T H\big], \quad y > 0. \tag{1.35}$$

If this dual problem admits a unique solution $Y_T^* \in \mathcal{Y}$, then the primal problem (1.2) also yields a unique solution

$$\begin{aligned} X_T^{\pi^*} &= x + \int_0^T X_s^{\pi^*} \pi_s^* \frac{\mathrm{d}\tilde{S}_s}{\tilde{S}_s} \\ &= x + \int_0^T \alpha_s^* \mathrm{d}S_s \\ &= I(yY_T^*) - H, \end{aligned}$$

with the corresponding optimal control $\pi^* = \frac{\alpha^* \tilde{S}}{X^{\pi^*}}$. Here, we have $I = (U')^{-1}$ and $x = -v'(y)$.[2]

The case of bounded terminal liability H is dealt with in Cvitanić et al. [34], where in a dual domain $\mathcal{Y}$ similar to (1.35) is employed. The case of general integrable H is studied in Hugonnier and Kramkov [63] by replacing the dual problem (1.35) by

$$v(y) = \inf_{Y_T \in \mathcal{Y}} \mathbb{E}\big[V(yY_T)\big], \quad y > 0,$$

but using a modification of the domain $\mathcal{Y}$. A ubiquitous property of the convex dual approach is that once the primal and the dual optimizers are obtained, their product $X^{\pi^*}Y^*$ is a non-negative martingale, see e.g. Kramkov and Schachermayer [80] for an economic interpretation. In the context of utility maximization with bounded random liabilities, this martingale property of $X^{\pi^*}Y^*$ is pointed out in Cvitanić et al. [34, Remark 4.6]. This martingale property of $X^{\pi^*}Y^*$ constitutes a first main ingredient for deriving a solution for the forward-backward equation (1.30). A second main ingredient is the structural characterization of the dual domain $\mathcal{Y}$. Note that in the setting of continuous processes, $\mathcal{Y}$ is the family of all non-negative supermartingales (see e.g. Kramkov

[2] This is equivalent to $u'(x) = y$ where $u(x) = \sup_\pi \mathbb{E}\big[U(X_T^\pi + H)\big]$. The differentiability of both v and u are shown in Cvitanić et al. [34].

and Schachermayer [80], Hugonnier and Kramkov [63]). According to a well-known result, every non-negative càdlàg supermartingale $Y \in \mathcal{Y}$ admits a unique multiplicative decomposition

$$Y = AM,$$

where A is a predictable, non-increasing process such that $A_0 = 1$ and M is a càdlàg local martingale. However, Larsen and Žitković [83] characterize the elements of $Y \in \mathcal{Y}$ by the multiplicative decomposition

$$Y = A\mathcal{E}(-\theta^{\mathcal{H}} \cdot W^{\mathcal{H}} + K \cdot W^{\mathcal{O}}), \tag{1.36}$$

where A is a predictable non-increasing process such that $A_0 = 1$ and $K \in \mathbb{H}^2_{loc}(\mathbb{R}^{d_2})$ (see Larsen and Žitković [83, Proposition 3.2]). Using that the Fenchel-Legendre transform V is strictly decreasing, Corollary 3.3 from Larsen and Žitković [83] shows that the dual optimizer is a (continuous) local martingale and admits the representation

$$Y^* = \mathcal{E}\big(- \theta^{\mathcal{H}} \cdot W^{\mathcal{H}} + K^* \cdot W^{\mathcal{O}}\big) \tag{1.37}$$

for a uniquely determined $K^* \in \mathbb{H}^2_{loc}(\mathbb{R}^{d_2})$. If $v(y) = \mathbb{E}\Big[V(yY^*_T)\Big] < \infty$, we can check that the optimal K^* actually belongs to $\mathbb{H}^2(\mathbb{R}^{d_2})$. This is done in the following result whose proof is of the same spirit as the one in Larsen [82, Lemma 3.2].

Lemma 1.4.1. *Assume that for $y > 0$, we have*

$$v(y) = \inf_{\nu\in\mathbb{H}^2_{loc}(\mathbb{R}^{d_2})} \mathbb{E}\Big[V\Big(y\mathcal{E}\big(- \theta^{\mathcal{H}} \cdot W^{\mathcal{H}} + \nu \cdot W^{\mathcal{O}}\big)_T\Big)\Big] < \infty.$$

Then, we also have

$$v(y) = \inf_{\nu\in\mathbb{H}^2(\mathbb{R}^{d_2})} \mathbb{E}\Big[V\Big(y\mathcal{E}\big(- \theta^{\mathcal{H}} \cdot W^{\mathcal{H}} + \nu \cdot W^{\mathcal{O}}\big)_T\Big)\Big],$$

i.e. the domain of minimization can be assumed to be $\mathbb{H}^2(\mathbb{R}^{d_2})$ instead of $\mathbb{H}^2_{loc}(\mathbb{R}^{d_2})$.

Proof. We consider the family of stopping times

$$\tau^n := \inf\Big\{t > 0 : \int_0^t \big(|\theta^{\mathcal{H}}_s|^2 + |K^*_s|^2\big)\mathrm{d}s \geq n\Big\},\ n \in \mathbb{N},$$

and we use the notation $\mathcal{E}_t(\cdot) := \mathcal{E}(\cdot)_t$. If $y > 0$, we have

$$\begin{aligned}
v(y) &= \mathbb{E}\Big[V\Big(y\mathcal{E}_T\big(- \theta^{\mathcal{H}} \cdot W^{\mathcal{H}} + K^* \cdot W^{\mathcal{O}}\big)\Big)\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[V\Big(y\mathcal{E}_T\big(- \theta^{\mathcal{H}} \cdot W^{\mathcal{H}} + K^* \cdot W^{\mathcal{O}}\big)\Big)\Big|\mathcal{F}_{\tau^n}\Big]\Big] \\
&\geq \mathbb{E}\Big[V\Big(y\mathcal{E}_{\tau^n}\big(- \theta^{\mathcal{H}} \cdot W^{\mathcal{H}} + K^* \cdot W^{\mathcal{O}}\big)\Big)\Big],
\end{aligned}$$

where the last line follows by Jensen's inequality. Continuing the last line and recalling that $V(y)$ is a strictly convex function, we have

$$\begin{aligned}
v(y) &\geq \mathbb{E}\Big[V\Big(y\exp\big(\int_0^{\tau^n}\big(-\theta_s^{\mathcal{H}}\mathrm{d}W_s^{\mathcal{H}} + K_s^*\mathrm{d}W_s^{\mathcal{O}}\big)\big)\exp\big(-\frac{1}{2}\int_0^{\tau^n}\big(|\theta_s^{\mathcal{H}}|^2 + |K_s^*|^2\big)\mathrm{d}s\big)\Big)\Big]\\
&\geq V\Big(\mathbb{E}\Big[y\exp\big(\int_0^{\tau^n}\big(-\theta_s^{\mathcal{H}}\mathrm{d}W_s^{\mathcal{H}} + K_s^*\mathrm{d}W_s^{\mathcal{O}}\big)\big)\exp\big(-\frac{1}{2}\int_0^{\tau^n}\big(|\theta_s^{\mathcal{H}}|^2 + |K_s^*|^2\big)\mathrm{d}s\big)\Big]\Big)\\
&\geq V\Big(y\exp\Big(\mathbb{E}\Big[-\frac{1}{2}\int_0^{\tau^n}\big(|\theta_s^{\mathcal{H}}|^2 + |K_s^*|^2\big)\mathrm{d}s\Big]\Big)\Big),
\end{aligned}$$

where Jensen's inequality has been used twice. By the continuity of V and the exponential function, it follows from the monotone convergence theorem that

$$\begin{aligned}
v(y) &\geq \lim_{n\to\infty} V\Big(\exp\Big(-\frac{1}{2}\mathbb{E}\Big[\int_0^{\tau^n}\big(|\theta_s^{\mathcal{H}}|^2 + |K_s^*|^2\big)\mathrm{d}s\Big]\Big)\Big)\\
&= V\Big(\exp\Big(-\frac{1}{2}\mathbb{E}\Big[\int_0^{T}\big(|\theta_s^{\mathcal{H}}|^2 + |K_s^*|^2\big)\mathrm{d}s\Big]\Big)\Big).
\end{aligned}$$

Since $v(y) < \infty$ and $V\big(\exp(-\infty)\big) = V(0) = U(\infty) = \infty$, it follows that

$$\mathbb{E}\Big[\int_0^T\big(|\theta_s^{\mathcal{H}}|^2 + |K_s^*|^2\big)\mathrm{d}s\Big] < \infty.$$

Taking into account that $\theta^{\mathcal{H}}$ is bounded, we deduce that $K^* \in \mathbb{H}^2(\mathbb{R}^{d_2})$. □

Now using that $X^{\pi^*}Y^*$ is a true martingale and that the dual optimizer Y^* is a local martingale satisfying (1.37), we get the following result.

Theorem 1.4.1. *Let H be a non-negative bounded random liability and assume that the coefficient of relative risk aversion $-\frac{xU''(x)}{U'(x)}$ satisfies*

$$\limsup_{x\to\infty}\left(-\frac{xU''(x)}{U'(x)}\right) < \infty. \tag{1.38}$$

Then there exists $x_0 > 0$ such that for all $x > x_0$ the coupled FBSDE (1.26) has a solution (X, Y, Z) satisfying $X_0 = x$. In addition, X is the optimal wealth process of the problem (1.2) and the dual optimizer Y^* associated with it is given by $Y^* = U'(X)\exp(Y)$ so that $Y_T^* = U'(X_T + H)$.

Proof. The existence of $x_0 > 0$ such that for every $x > x_0$ the quantity

$$u(x) = \sup_{\pi\in\Pi^x}\mathbb{E}\big[U\big(X_T^\pi + H\big)\big] = \mathbb{E}\big[U\big(X_T^{\pi^*} + H\big)\big]$$

is finite has been shown Cvitanić et al. [34]. We set $X^* := X^{\pi^*}$. We also recall from Cvitanić et al. [34] that $y = u'(x) > 0$ for $x > x_0$ and

$$\mathbb{E}\Big[yX_T^*Y_T^*\Big] = xy.$$

Since $\alpha := yX^*Y^*$ is a true martingale, by the predictable representation property for Brownian martingales, we have

$$\alpha_t = \mathbb{E}\Big[yX_T^*Y_T^*|\mathcal{F}_t\Big] = xy + \int_0^t \beta_s \mathrm{d}W_s, \quad t \in [0,T],$$

with $\beta \in \mathbb{H}^2_{loc}(\mathbb{R}^d)$. We now define a semimartingale Y (which will be the solution to the backward component of the FBSDE (1.26)) via

$$\alpha = yX^*Y^* = X^*U'(X^*)\exp(Y) \tag{1.39}$$

such that $\alpha_T = X^*U'(X_T^* + H)$, i.e.

$$\begin{aligned} Y &= \log\alpha - \log X^* - \log U'(X^*), \\ Y_T &= \log\Big(\frac{yY_T^*}{U'(X_T^*)}\Big) = \log\Big(\frac{U'(X_T^*+H)}{U'(X_T^*)}\Big). \end{aligned}$$

An application of Itô's formula to $Y = \log\alpha - \log X^* - \log U'(X^*)$ shows that

$$\begin{aligned} Y_t &= \log\left(\frac{U'(X_T^*+H)}{U'(X_T^*)}\right) - \int_t^T \Big(\frac{\beta_s^{\mathcal{H}}}{\alpha_s} - \pi_s^* - \frac{U''(X_s^*)}{U'(X_s^*)}\pi_s^*X_s^*\Big)\mathrm{d}W_t^{\mathcal{H}} - \int_t^T \frac{\beta_s^{\mathcal{O}}}{\alpha_s}\mathrm{d}W_s^{\mathcal{O}} \\ &\quad - \int_t^T \Big[-\frac{1}{2}\frac{|\beta_s|^2}{|\alpha_s|^2} + \frac{1}{2}|\pi_s^*|^2 - \Big(\frac{U''(X_s^*)}{U'(X_s^*)}X_s^* + 1\Big)\pi_s^*\theta_s^{\mathcal{H}} \\ &\quad - \frac{1}{2}\Big(\frac{U^{(3)}(X_s^*)}{U'(X_s^*)} - \Big|\frac{U''(X_s^*)}{U'(X_s^*)}\Big|^2\Big)|\pi_s^*X_s^*|^2\Big]\mathrm{d}s \\ &= Y_T - \int_t^T (K_s^{\mathcal{H}} + K_s^{\mathcal{O}})\mathrm{d}W_s - \int_t^T f_s\mathrm{d}s, \end{aligned} \tag{1.40}$$

where we have

$$K_t^{\mathcal{H}} := \frac{\beta_t^{\mathcal{H}}}{\alpha_t} - \pi_t^* - \frac{U''(X_t^*)}{U'(X_t^*)}\pi_t^*X_t^*, \;\; K_t^{\mathcal{O}} := \frac{\beta_t^{\mathcal{O}}}{\alpha_t},$$

and

$$f_t := -\frac{1}{2}\frac{|\beta_t|^2}{|\alpha_t|^2} + \frac{1}{2}|\pi_t^*|^2 - \Big(\frac{U''(X_t^*)}{U'(X_t^*)}X_t^* + 1\Big)\pi_t^*\theta_t^{\mathcal{H}} - \frac{1}{2}\Big(\frac{U^{(3)}(X_t^*)}{U'(X_t^*)} - \Big|\frac{U''(X_t^*)}{U'(X_t^*)}\Big|^2\Big)|\pi_t^*X_t^*|^2.$$

This shows that Y is continuous. However, from (1.39) we have $Y = \log\Big(\frac{yY^*}{U'(X^*)}\Big)$. Thus applying Itô's formula once again to $Y = \log\Big(\frac{yY^*}{U'(X^*)}\Big)$, where the dynamics of Y^* is governed by (1.37), we get

$$Y_t = \log\left(\frac{U'(X_T^* + H)}{U'(X_T^*)}\right) - \int_t^T -\left(\theta_s^{\mathcal{H}} + \frac{U''(X_s^*)X_s^*}{U'(X_s^*)}\pi_s^*\right)\mathrm{d}W_s^{\mathcal{H}} - \int_t^T K_s^* \mathrm{d}W_s^{\mathcal{O}}$$
$$- \int_t^T \Big(-\frac{1}{2}|\theta_s^{\mathcal{H}}|^2 - \frac{U''(X_s^*)X_s^*}{U'(X_s^*)}\pi_s^*\theta_s^{\mathcal{H}} - \frac{1}{2}\frac{U^{(3)}(X_s^*)}{U'(X_s^*)}\big|\pi_s^* X_s^*\big|^2$$
$$+ \frac{1}{2}\Big|\frac{U''(X_s^*)}{U'(X_s^*)}\Big|^2 \big|\pi_s^* X_s^*\big|^2 - \frac{1}{2}|K_s^*|^2\Big)\mathrm{d}s. \tag{1.41}$$

Since (1.40) and (1.41) must coincide, we get by comparing the integrands of the $\mathrm{d}W^{\mathcal{H}}$ and $\mathrm{d}W^{\mathcal{O}}$ expressions

$$K^{\mathcal{O}} = K^*, \quad \frac{\beta^{\mathcal{H}}}{\alpha} - \pi^* - \frac{U''(X^*)}{U'(X^*)}\pi^* X^* = \left(-\theta^{\mathcal{H}} + \frac{U''(X_s^*)}{U'(X_s^*)}\pi^* X^*\right).$$

This now yields

$$\pi^* = \frac{\beta^{\mathcal{H}}}{\alpha} + \theta^{\mathcal{H}}. \tag{1.42}$$

Recalling $\mathrm{d}X_t^* = \pi_t^* X_t^* \big(\mathrm{d}W_t^{\mathcal{H}} + \theta_t^{\mathcal{H}}\mathrm{d}t\big)$ and applying Itô's formula to $\exp(Y)U'(X^*)$, we obtain

$$\mathrm{d}\Big(e^{Y_t}U'(X_t^*)\Big) = e^{Y_t}\Big(U''(X_t^*)\pi_t^* X_t^* + U'(X_t^*)K_t^{\mathcal{H}}\Big)\mathrm{d}W_t^{\mathcal{H}} + e^{Y_t}U'(X_t^*)K_t^{\mathcal{O}}\mathrm{d}W_t^{\mathcal{O}}$$
$$+ e^{Y_t}\Big(U''(X_t^*)\pi_t^* X_t^* \theta_t^{\mathcal{H}} + \frac{1}{2}U^{(3)}(X_t^*)|\pi_t^* X_t^*|^2 + U'(X_t^*)f_t$$
$$+ \frac{1}{2}U'(X_t^*)|K_t^{\mathcal{H}}|^2 + \frac{1}{2}U'(X_t^*)|K_t^{\mathcal{O}}|^2 + K_t^{\mathcal{H}}U''(X_t^*)\pi_t^* X_t^*\Big)\mathrm{d}t.$$

Making use of (1.40), the above equation simplifies to

$$\mathrm{d}\Big(e^{Y_t}U'(X_t^*)\Big) = e^{Y_t}U'(X_t^*)\Big(- \pi_t^*\theta_t^{\mathcal{H}} + |\pi_t^*|^2 - \frac{\beta_t^{\mathcal{H}}}{\alpha_t}\pi_t^*\Big)\mathrm{d}t$$
$$+ e^{Y_t}U'(X_t^*)\Big(\frac{\beta_t^{\mathcal{H}}}{\alpha_t} - \pi_t^*\Big)\mathrm{d}W_t^{\mathcal{H}} + e^{Y_t}U'(X_t^*)\frac{\beta_t^{\mathcal{O}}}{\alpha_t}\mathrm{d}W_t^{\mathcal{O}}.$$

Now plugging (1.42) into the equation above, we get

$$\mathrm{d}\Big(e^{Y_t}U'(X_t^*)\Big) = e^{Y_t}U'(X_t^*)\Big(- \theta_t^{\mathcal{H}}\mathrm{d}W_t^{\mathcal{H}}\Big) + e^{Y_t}U'(X_t^*)\frac{\beta_t^{\mathcal{O}}}{\alpha_t}\mathrm{d}W_t^{\mathcal{O}},$$

and it follows from $yY^* = e^Y U'(X^*)$ (see (1.39)) that $K^* = \frac{\beta^{\mathcal{O}}}{\alpha} \in \mathbb{H}^2(\mathbb{R}^{d_2})$. In analogy to the proof of Theorem 1.3.1, once we define

$$Z^{\mathcal{H}} := K^{\mathcal{H}} = \frac{\beta^{\mathcal{H}}}{\alpha} - \pi^* - \frac{U''(X^*)}{U'(X^*)}\pi^* X^* = -\frac{U''(X^*)}{U'(X^*)}\Big(\frac{\beta^{\mathcal{H}}}{\alpha} + \theta^{\mathcal{H}}\Big)X^* - \theta^{\mathcal{H}},$$
$$Z^{\mathcal{O}} := K^{\mathcal{O}} = \frac{\beta^{\mathcal{O}}}{\alpha},$$

equation (1.40) attains the form of the BSDE (1.26), i.e.

$$\begin{aligned} Y_t = Y_T &- \int_t^T Z_s^{\mathcal{H}} \mathrm{d}W_s^{\mathcal{H}} - \int_t^T Z_s^{\mathcal{O}} \mathrm{d}W_s^{\mathcal{O}} \\ &- \int_t^T \Bigg(|Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}|^2\Big(1 - \frac{1}{2}\frac{U^{(3)}(X_s^*)U'(X_s^*)}{U^{('')}(X_s^*)^2}\Big) - \frac{1}{2}|Z_s^{\mathcal{H}}|^2 - \frac{1}{2}|Z_s^{\mathcal{O}}|^2\Bigg)\mathrm{d}s. \end{aligned} \tag{1.43}$$

Now together with

$$\begin{aligned} X_t^* &= x + \int_0^t \pi_s^* X_s^* \left(\mathrm{d}W_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}\mathrm{d}s\right) \\ &= x + \int_0^t -\frac{U'(X_s^*)}{U''(X_s^*)X_s^*}\Big(Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}\Big) \cdot X_s^* \left(\mathrm{d}W_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}\mathrm{d}s\right) \\ &= x - \int_0^t \frac{U'(X_s^*)}{U''(X_s^*)}\Big(Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}\Big)\mathrm{d}W_s^{\mathcal{H}} - \int_0^t \frac{U'(X_s^*)}{U''(X_s^*)}\Big(Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}\Big)\theta_s^{\mathcal{H}}\mathrm{d}s \end{aligned} \tag{1.44}$$

this verifies that $((X^*, Y, Z)$ is a solution to the FBSDE (1.26). Finally, due to $K^* = K^{\mathcal{O}} = Z^{\mathcal{O}}$, we get that

$$Y^* = yU'(X)\exp(Y).$$

□

Let us recall that the *absolute risk aversion* of $U(x)$ is defined as $ARA(x) := -\frac{U''(x)}{U'(x)}$ and the *risk tolerance* as $\frac{1}{ARA(x)}$. We say that $U(x)$ has *hyperbolic absolute risk aversion (HARA)* if and only if its risk tolerance $\frac{1}{ARA(x)}$ is linear in x. More precisely, it can be shown that a utility function $U(x), x \geq 0$, is *HARA* if and only if for given constants $\gamma, a, b \in \mathbb{R}$ such that $a > 0$ and $\frac{ax}{1-\gamma} + b > 0$, we have

$$U(x) = \frac{1-\gamma}{\gamma}\Big(\frac{ax}{1-\gamma} + b\Big)^{\gamma},$$

Corollary 1.4.1. *Assume that U is HARA. Then there exists a constant $\kappa \in \mathbb{R}$ such that the backward equation from* (1.26) *can be written as*

$$\begin{aligned} Y_t &= \log\Big(\frac{U'(X_T^* + H)}{U'(X_T^*)}\Big) - \int_t^T Z_s \mathrm{d}W_s - \int_t^T \Big(-\frac{1}{2}|Z_s|^2 + \kappa|Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}|^2\Big)\mathrm{d}s \\ &= \log\Big(\frac{U'(X_T^*+H)}{U'(X_T^*)}\Big) - \int_t^T Z_s \mathrm{d}W_s - \int_t^T g(s, Z_s)\mathrm{d}s. \end{aligned} \tag{1.45}$$

Proof. Notice that for the risk tolerance

$$f(x) := \frac{1}{ARA(x)} = -\frac{U'(x)}{U''(x)}$$

it holds that

$$f'(x) = -1 + \frac{U'(x)U^{(3)}(x)}{|U''(x)|^2}.$$

Since U being *HARA* implies that f is linear in x, it follows that there exist constants $c, d \in \mathbb{R}$ such that $f(x) = cx + d$. Hence the BSDE from (1.26) can also be written as

$$\begin{aligned} Y_t &= \log\Big(\frac{U'(X_T^* + H)}{U'(X_T^*)}\Big) - \int_t^T Z_s \mathrm{d}W_s - \int_t^T \Big(-\frac{1}{2}|Z_s|^2 + \big(\frac{1}{2} - \frac{1}{2}f'(X_s^*)\big)|Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}|^2\Big)\mathrm{d}s \\ &= \log\Big(\frac{U'(X_T^* + H)}{U'(X_T^*)}\Big) - \int_t^T Z_s \mathrm{d}W_s - \int_t^T \Big(-\frac{1}{2}|Z_s|^2 + \kappa|Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}|^2\Big)\mathrm{d}s, \end{aligned}$$

for $\kappa = \frac{1}{2} - \frac{1}{2}c$. □

Note that for the power utility function $U(x) = \frac{1}{p}x^p$ with $p \in (0,1)$, the FBSDE becomes

$$\begin{cases} X_t = x + \int_0^t \frac{X_s(Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}})}{1-p}\mathrm{d}W_s^{\mathcal{H}} + \int_0^t \theta_s^{\mathcal{H}} \frac{X_s(Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}})}{1-p}\mathrm{d}s, \\ Y_t = (p-1)\log\Big(1 + \frac{H}{X_T^{\pi^*}}\Big) - \int_t^T Z_s \mathrm{d}W_s - \int_t^T \frac{p}{2(p-1)}|Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}|^2 - \frac{1}{2}|Z_s|^2 \mathrm{d}s. \end{cases} \tag{1.46}$$

According to Theorem 1.4.1, this FBSDE admits a solution whose construction however is based on the knowledge of the solution to the convex dual problem. In the next chapter, we attack the solution of (1.46) by an approach which does not rely on the existence of the dual optimizer but rather on a compactness result for martingales which tackles the utility optimization problem directly. In this respect, Chapter 2 solves (1.46) by an alternative, self-contained ansatz.

2 Coupled FBSDEs for power utility maximization with endowment

In this chapter, we present a method to solve the fully coupled FBSDE equation with quadratic growth (1.46) from Chapter 1. A simple transformation merges both forward and backward components to a single backward equation without drift, thus a local martingale with a terminal condition. Choosing a suitable maximizing sequence of terminal wealths and successively applying the predictable representation property together with some useful estimates then allows to employ a result from Delbaen and Schachermayer [39] on the compactness of bounded sequences of martingales in $\mathcal{H}^1$. This equips us with a limit candidate. Using this candidate, we then strip the corresponding forward and backward equations. By construction, the forward process coincides with the optimal wealth process. However, to show that this pair indeed gives rise to the solution of the FBSDE, we identify the backward equation as the adjoint equation that is associated to the Hamiltonian system of the power utility maximization problem.

2.1 Conditions on the parameters

The power utility function is given by $U(x) = \frac{1}{p}x^p, x > 0$, for some constant $p \in (0,1)$. We slightly deviate from the notation of Section 1.1 and write $W = (W^{\mathcal{H}}, W^{\mathcal{O}})$ for a D-dimensional standard Brownian motion where we denote $W^{\mathcal{H}} = (W^1, \dots, W^{d_1})$ as the first d_1 components and $W^{\mathcal{O}} = (W^{d_1+1}, \dots, W^d)$ the remaining $d - d_1$ components of the Brownian motion. To reproduce the setting of Section 1.1 further, we denote θ as the market price of risk which is a d-dimensional adapted process for which we also adopt the notation $\theta^{\mathcal{H}} = (\theta^1, \dots, \theta^{d_1})$ and $\theta^{\mathcal{O}} = (\theta^{d_1+1}, \dots, \theta^d)$. Moreover, we denote by $\mathbb{H}^2_{loc}(\mathbb{R}^m)$ as the space of all predictable processes taking values in $\mathbb{R}^m$ that are locally square integrable on $[0,T]$. If the range $\mathbb{R}^m$ is clear from the context, we will also omit it and simply write $\mathbb{H}^2_{loc}$. We moreover consider a bounded random variable $H \in L^\infty(\Omega, \mathcal{F}_T)$ as the terminal endowment. Throughout this chapter, we make the following assumptions.

(H1) $\theta = (\theta^{\mathcal{H}}, \theta^{\mathcal{O}})$ is uniformly bounded;

(H2) there exist constants $0 < c < C$ such that the $\mathcal{F}_T$-measurable random variable H satisfies $c \leq H \leq C$ a.s.

The central object of our study is equation (1.46),

$$X_t = x + \int_0^t X_s \left(\frac{1}{1-p} \big(Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}} \big) \right) \mathrm{d}W_s^{\mathcal{H}} + \int_0^t X_s \left(\frac{1}{1-p} \big(Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}} \big) \theta_s^{\mathcal{H}} \right) \mathrm{d}s, \tag{2.1}$$

$$\begin{aligned} Y_t = \log \left(\frac{X_T + H}{X_T} \right)^{p-1} &- \int_t^T Z_s^{\mathcal{H}} \mathrm{d}W_s^{\mathcal{H}} - \int_t^T Z_s^{\mathcal{O}} \mathrm{d}W_s^{\mathcal{O}} + \frac{1}{2} \int_t^T |Z_s^{\mathcal{O}}|^2 \mathrm{d}s \\ &- \int_t^T \left(\frac{p}{2(p-1)} |Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}|^2 - \frac{1}{2} |Z_s^{\mathcal{H}}|^2 \right) \mathrm{d}s. \end{aligned} \tag{2.2}$$

Our aim is to show an existence result for this FBSDE in a self-contained way which does not rely e.g. on the assumptions and results from convex duality theory (see Chapter 1.4.2). Note that the $\mathbb{R}^d$-valued martingale integrand Z splits here into $Z = (Z^{\mathcal{H}}, Z^{\mathcal{O}})$ with $Z^{\mathcal{H}} = (Z^1, \dots, Z^{d_1})$ and $Z^{\mathcal{O}} = (Z^{d_1+1}, \dots, Z^d)$.

2.2 Martingale transformations

This section provides a simple yet instructive transformation which allows to merge the system (2.1), (2.2) into one single driftless equation with a terminal condition. It plays an important role in Section 2.3 where a solution to the FBSDE (2.1), (2.2) is constructed.

Lemma 2.2.1. *Assume that the triplet (X, Y, Z) of adapted processes is a solution to the FBSDE (2.1), (2.2). Then, for $U(x) = \frac{1}{p}x^p$ with $p \in (0,1)$, the process $P := XU'(X)e^Y$ is a local martingale which satisfies the equation*

$$P_t = U'(X_T + H) X_T - \int_t^T \frac{1}{1-p} P_s \big(Z_s^{\mathcal{H}} + p\theta_s^{\mathcal{H}} \big) \mathrm{d}W_s^{\mathcal{H}} - \int_t^T P_s Z_s^{\mathcal{O}} \mathrm{d}W_s^{\mathcal{O}}. \tag{2.3}$$

Proof. By $U'(x) = x^{p-1}$, we have $P_t = X_t U'(X_t) e^{Y_t} = X_t^p e^{Y_t}$. First, we have

$$\begin{aligned} \mathrm{d}X_t^p &= pX_t^{p-1} X_t \frac{Z_t^{\mathcal{H}} + \theta_t^{\mathcal{H}}}{1-p} \left(\mathrm{d}W_t^{\mathcal{H}} + \theta_t^{\mathcal{H}} \right) + \frac{1}{2} p(p-1) X_t^{p-2} X_t^2 \left| \frac{Z_t^{\mathcal{H}} + \theta_t^{\mathcal{H}}}{1-p} \right|^2 \mathrm{d}t \\ &= X_t^p \left[p \frac{Z_t^{\mathcal{H}} + \theta_t^{\mathcal{H}}}{1-p} \mathrm{d}W_t^{\mathcal{H}} + \left\{ p \frac{Z_t^{\mathcal{H}} + \theta_t^{\mathcal{H}}}{1-p} \theta_t^{\mathcal{H}} + \frac{1}{2} p(p-1) X_t^{p-2} X_t^2 \left| \frac{Z_t^{\mathcal{H}} + \theta_t^{\mathcal{H}}}{1-p} \right|^2 \right\} \mathrm{d}t \right] \\ &= \frac{p}{1-p} X_t^p \left[\big(Z_t^{\mathcal{H}} + \theta_t^{\mathcal{H}} \big) \mathrm{d}W_t^{\mathcal{H}} + \left\{ (Z_t^{\mathcal{H}} + \theta_t^{\mathcal{H}}) \theta_t^{\mathcal{H}} - \frac{1}{2} |Z_t^{\mathcal{H}} + \theta_t^{\mathcal{H}}|^2 \right\} \mathrm{d}t \right]. \end{aligned} \tag{2.4}$$

Moreover, we see that

$$\begin{aligned} \mathrm{d}e^{Y_t} &= e^{Y_t} \left(Z_t^{\mathcal{H}} \mathrm{d}W_t^{\mathcal{H}} + Z_t^{\mathcal{O}} \mathrm{d}W_t^{\mathcal{O}} - \frac{1}{2} |Z_t^{\mathcal{O}}|^2 \mathrm{d}t - \frac{1}{2} \Big(\frac{p}{1-p} |Z_t^{\mathcal{H}} + \theta_t^{\mathcal{H}}|^2 + |Z_t^{\mathcal{H}}|^2 \Big) \mathrm{d}t \right) \\ &\quad + \frac{1}{2} e^{Y_t} \Big(|Z_t^{\mathcal{H}}|^2 \mathrm{d}t + |Z_t^{\mathcal{O}}|^2 \mathrm{d}t \Big) \\ &= e^{Y_t} \left(Z_t^{\mathcal{H}} \mathrm{d}W_t^{\mathcal{H}} + Z_t^{\mathcal{O}} \mathrm{d}W_t^{\mathcal{O}} - \frac{1}{2(1-p)} |Z_t^{\mathcal{H}} + \theta_t^{\mathcal{H}}|^2 \mathrm{d}t \right), \end{aligned} \tag{2.5}$$

from which we deduce

$$
\begin{aligned}
\mathrm{d}P_t &= P_t\Bigg[\Big(Z_t^{\mathcal{H}} + \frac{p}{1-p}\big(Z_t^{\mathcal{H}} + \theta_t^{\mathcal{H}}\big)\Big)\mathrm{d}W_t^{\mathcal{H}} + Z_t^{\mathcal{O}}\mathrm{d}W_t^{\mathcal{O}} \\
&\qquad + \underbrace{\Big(-\frac{p}{2(1-p)}|Z_t^{\mathcal{H}} + \theta_t^{\mathcal{H}}|^2 - \frac{p}{2(1-p)}|Z_t^{\mathcal{H}} + \theta_t^{\mathcal{H}}|^2 + \frac{p}{1-p}|Z_t^{\mathcal{H}} + \theta_t^{\mathcal{H}}|^2\Big)\mathrm{d}t}_{=0}\Bigg] \\
&= P_t\left(\frac{1}{1-p}(Z_t^{\mathcal{H}} + p\theta_t^{\mathcal{H}})\mathrm{d}W_t^{\mathcal{H}} + Z_t^{\mathcal{O}}\mathrm{d}W_t^{\mathcal{O}}\right).
\end{aligned}
\tag{2.6}
$$

Note that we obtain from (2.2)

$$
P_T = X_T U'(X_T)e^{Y_T} = X_T U'(X_T)\frac{U'(X_T + H)}{U'(X_T)} = U'(X_T + H)X_T.
$$

Hence, P is a local martingale with terminal condition $P_T = U'(X_T + H)X_T$. □

As a straightforward consequence of Lemma 2.2.1 we obtain the following moment estimate for the forward process X. A similar result plays a key role in the construction of a solution to the FBSDE, see Lemma 2.3.1.

Proposition 2.2.1. *Let $U(x) = \frac{1}{p}x^p$ with $p \in (0,1)$. For $Z \in \mathbb{H}^2_{loc}(\mathbb{R}^{d_1})$ let X be*

$$
X_t = x + \int_0^t X_s\left(\frac{1}{1-p}\big(Z_s + \theta_s^{\mathcal{H}}\big)\right)\mathrm{d}W_s^{\mathcal{H}} + \int_0^t X_s\left(\frac{1}{1-p}\big(Z_s + \theta_s^{\mathcal{H}}\big)\theta_s^{\mathcal{H}}\right)\mathrm{d}s.
$$

There exists a constant $C > 0$ such that

$$
\sup_{0\le t\le T}\mathbb{E}|X_t|^p \le Cx^p, \tag{2.7}
$$

where the constant C is bounded by

$$
C \le \mathbb{E}\left[\exp\left(\int_0^T \frac{p}{2(1-p)}|\theta_s^{\mathcal{H}}|^2\mathrm{d}s\right)\right].
$$

Proof. According to (2.4), we have

$$
\begin{aligned}
X_t^p = x^p \exp&\left(\int_0^t \frac{p}{1-p}(Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}})\mathrm{d}W_s^{\mathcal{H}} - \frac{1}{2}\int_0^t \Big|\frac{p}{1-p}(Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}})\Big|^2\mathrm{d}s\right) \\
&\times \exp\left(\int_0^t \frac{p}{2(1-p)}\big(|\theta_s^{\mathcal{H}}|^2 - |Z_s^{\mathcal{H}}|^2\big)\mathrm{d}s\right),
\end{aligned}
$$

which we can bound from above by

$$
\begin{aligned}
X_t^p &\leq x^p \mathcal{E}\left(\frac{p}{1-p}(Z^{\mathcal{H}}+\theta^{\mathcal{H}}) * W^{\mathcal{H}}\right)_t \exp\left(-\int_0^t |Z_s^{\mathcal{H}}|^2 \mathrm{d}s\right) \exp\left(\int_0^T \frac{p}{2(1-p)}|\theta_s^{\mathcal{H}}|^2 \mathrm{d}s\right) \\
&\leq C x^p \mathcal{E}\left(\frac{p}{1-p}(Z^{\mathcal{H}}+\theta^{\mathcal{H}}) * W^{\mathcal{H}}\right)_t,
\end{aligned}
$$

where $\mathcal{E}(\cdot)$ denotes the stochastic exponential. Since $\mathcal{E}\left(\frac{p}{1-p}(Z^{\mathcal{H}}+\theta^{\mathcal{H}}) * W^{\mathcal{H}}\right)$ is a non-negative local martingale, hence a supermartingale, we get

$$
\mathbb{E}X_t^p \leq C x^p, \quad 0 \leq t \leq T.
$$

□

Another transformation which becomes useful for the construction of the backward equation in Theorem 2.3.2 is the following result.

Lemma 2.2.2. *For $p \in (0,1)$ and $U(x) = \frac{1}{p}x^p$, let $(X, P, Z^{\mathcal{H}}, Z^{\mathcal{O}})$ be a solution of the coupled FBSDE*

$$
\begin{aligned}
X_t &= x + \int_0^t X_s \left(\frac{1}{1-p}\big(Z_s^{\mathcal{H}}+\theta_s^{\mathcal{H}}\big)\right) \mathrm{d}W_s^{\mathcal{H}} + \int_0^t X_s \left(\frac{1}{1-p}\big(Z_s^{\mathcal{H}}+\theta_s^{\mathcal{H}}\big)\theta_s^{\mathcal{H}}\right) \mathrm{d}s, \\
P_t &= U'(X_T+H)X_T - \int_t^T \frac{1}{1-p} P_s \big(Z_s^{\mathcal{H}} + p\theta_s^{\mathcal{H}}\big) \mathrm{d}W_s^{\mathcal{H}} - \int_t^T P_s Z_s^{\mathcal{O}} \mathrm{d}W_s^{\mathcal{O}},
\end{aligned}
$$

where $t \in [0,T]$. Then, the process $Y := \log P - p \log X$ satisfies the BSDE (2.2).

Proof. Using the fact that $\log P_T - p \log X_T = \log \frac{P_T}{X_T^p} = \log \frac{U'(X_T+H)X_T}{U'(X_T)X_T} = \log \frac{U'(X_T+H)}{U'(X_T)}$, we have $Y_T = \log \frac{U'(X_T+H)}{U'(X_T)}$. Now Itô's formula yields

$$
\begin{aligned}
\mathrm{d}Y_t &= P_t^{-1}\mathrm{d}P_t - \frac{1}{2}P_t^{-2}\mathrm{d}\langle P\rangle_t - pX_t^{-1}\mathrm{d}X_t + \frac{1}{2}pX_t^{-2}\mathrm{d}\langle X\rangle_t \\
&= \left(\frac{1}{1-p}\big(Z_t^{\mathcal{H}} + p\theta_t^{\mathcal{H}}\big) - \frac{p}{1-p}\big(Z_t^{\mathcal{H}}+\theta_t^{\mathcal{H}}\big)\right)\mathrm{d}W_t^{\mathcal{H}} + Z_t^{\mathcal{O}}\mathrm{d}W_t^{\mathcal{O}} - \frac{1}{2}|Z_t^{\mathcal{O}}|^2\mathrm{d}t \\
&\quad + \left(\frac{p}{2(1-p)^2}\big|Z_t^{\mathcal{H}}+\theta_t^{\mathcal{H}}\big|^2 - \frac{p}{1-p}(Z_t^{\mathcal{H}}+\theta_t^{\mathcal{H}})\theta_t^{\mathcal{H}} - \frac{1}{2(1-p)^2}\big|Z_t^{\mathcal{H}}+p\theta_t^{\mathcal{H}}\big|^2\right)\mathrm{d}t.
\end{aligned}
$$

Note that we have

$$
\begin{aligned}
\frac{p}{2(1-p)^2}\big|Z_t^{\mathcal{H}}+\theta_t^{\mathcal{H}}\big|^2 - \frac{1}{2(1-p)^2}\big|Z_t^{\mathcal{H}}+p\theta_t^{\mathcal{H}}\big|^2 &= \frac{1}{2(1-p)} \cdot \frac{(p-1)|Z_t^{\mathcal{H}}|^2 + p(1-p)|\theta_t^{\mathcal{H}}|^2}{1-p} \\
&= \frac{1}{2(1-p)}\Big(-|Z_t^{\mathcal{H}}|^2 + p|\theta_t^{\mathcal{H}}|^2\Big) = \frac{1}{2(1-p)}\Big(p|\theta_t^{\mathcal{H}}|^2 - p|Z_t^{\mathcal{H}}|^2 + p|Z_t^{\mathcal{H}}|^2 - |Z_t^{\mathcal{H}}|^2\Big) \\
&= \frac{p}{2(1-p)}\Big(|\theta_t^{\mathcal{H}}|^2 - |Z_t^{\mathcal{H}}|^2\Big) - \frac{1}{2}|Z_t^{\mathcal{H}}|^2,
\end{aligned}
$$

hence, we get

$$
\begin{aligned}
&\frac{p}{2(1-p)^2}\big|Z^{\mathcal{H}}_t+\theta^{\mathcal{H}}_t\big|^2-\frac{p}{1-p}(Z^{\mathcal{H}}_t+\theta^{\mathcal{H}}_t)\theta^{\mathcal{H}}_t-\frac{1}{2(1-p)^2}\big|Z^{\mathcal{H}}_t+p\theta^{\mathcal{H}}_t\big|^2\\
&\quad=-\frac{1}{2}|Z^{\mathcal{H}}_t|^2+\frac{p}{2(1-p)}\Big(|\theta^{\mathcal{H}}_t|^2-|Z^{\mathcal{H}}_t|^2\Big)-\frac{p}{1-p}(Z^{\mathcal{H}}_t+\theta^{\mathcal{H}}_t)\theta^{\mathcal{H}}_t\\
&\quad=-\frac{1}{2}|Z^{\mathcal{H}}_t|^2+\frac{p}{2(1-p)}\Big(|\theta^{\mathcal{H}}_t|^2-|Z^{\mathcal{H}}_t|^2-2(Z^{\mathcal{H}}_t+\theta^{\mathcal{H}}_t)\theta^{\mathcal{H}}_t\Big)\\
&\quad=-\frac{1}{2}|Z^{\mathcal{H}}_t|^2+\frac{p}{2(1-p)}(Z^{\mathcal{H}}_t+\theta^{\mathcal{H}}_t)\cdot(-1)(Z^{\mathcal{H}}_t+\theta^{\mathcal{H}}_t)\\
&\quad=-\frac{1}{2}|Z^{\mathcal{H}}_t|^2+\frac{p}{2(p-1)}|Z^{\mathcal{H}}_t+\theta^{\mathcal{H}}_t|^2.
\end{aligned}
$$

This gives rise to

$$
\mathrm{d}Y_t=Z^{\mathcal{H}}_t\mathrm{d}W^{\mathcal{H}}_t+Z^{\mathcal{O}}_t\mathrm{d}W^{\mathcal{O}}_t-\frac{1}{2}|Z^{\mathcal{O}}_t|^2\mathrm{d}t+\left(\frac{p}{2(p-1)}\big|Z^{\mathcal{H}}_t+\theta^{\mathcal{H}}_t\big|^2-\frac{1}{2}|Z^{\mathcal{H}}_t|^2\right)\mathrm{d}t,
$$

which finishes the proof. □

2.3 Solving the FBSDE

The focus of this section is to establish a solution for the FBSDE (2.1), (2.2). Our approach is based on directly providing a solution to the underlying utility maximization problem from Chapter 1. We proceed in two steps: first, we solve in a primal approach for an optimal control process π and second, we strip from the existence of this optimal control process the existence of the FBSDE (X,Y,Z). To establish the link to utility maximization problems, let us denote by

$$
\begin{aligned}
\mathcal{C}(x)=\Big\{X_T\in\mathcal{F}_T:\ \exists\pi\in\mathbb{H}^2_{loc}(\mathbb{R}^{d_1})\text{ s.t. }X_t=x+\int_0^T X_t\pi_t\big(\mathrm{d}W^{\mathcal{H}}_t+\theta^{\mathcal{H}}_t\mathrm{d}t\big)\\
\text{and }\mathbb{E}\big[U(X_T+H)\big]<\infty\ \Big\}
\end{aligned}
\tag{2.8}
$$

the set of all *admissible* terminal wealths. Let us also define the quantities

$$
v(x):=\sup_{X_T\in\mathcal{C}(x)}\mathbb{E}\big[U\big(X_T+H\big)\big],\ w(x):=\sup_{X_T\in\mathcal{C}(x)}\mathbb{E}\big[X_TU'\big(X_T+H\big)\big].\tag{2.9}
$$

Note that $v(x)$ is the optimal expected utility from terminal wealth in the presence of the random endowment H. Investments start with an initial capital $x>0$ and the problem of solving $v(x)$ coincides with the utility optimization problem (1.2) from Chapter 1. However, note that the optimization for $w(x)$ does not solve for the optimal terminal utility but rather solves for a weighted optimal marginal utility. We see in Section 2.4 why solving for this kind of marginal utility is sufficient to obtain a solution to the primal problem $v(x)$. Loosely speaking, $w(x)$ constitutes a first order sufficient optimality criterion for $v(x)$ being optimal. We shall give more details on this in Section 2.4. Let us start with a technical remark which is important for the proof of Theorem 2.3.1.

Remark 2.3.1. *Let us assume that for* $\pi \in \mathbb{H}^2_{loc}$ *we have* $X_t := x + \int_0^t \pi_s(\mathrm{d}W_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}\mathrm{d}s) > 0$ *for every* $t \in [0,T]$*. Then there exists a* $\widehat{\pi} \in \mathbb{H}^2_{loc}$ *such that*

$$X_T = x + \int_0^T X_s \widehat{\pi}_s\big(\mathrm{d}W_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}\mathrm{d}s\big).$$

We see this because we trivially have

$$X_T = x + \int_0^t \pi_s\big(\mathrm{d}W_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}\mathrm{d}s\big) = x + \int_0^t X_s \frac{\pi_s}{X_s}\big(\mathrm{d}W_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}\mathrm{d}s\big), \quad 0 \le t \le T.$$

Since θ *is bounded, we have by the Girsanov theorem that* $\mathrm{d}W^\theta := \mathrm{d}W_t + \theta_t \mathrm{d}t$ *is a Brownian motion under a probability measure* $\mathbb{P}^\theta \sim \mathbb{P}$*. Defining* $\hat{\pi}_t := \frac{\pi_t}{X_t}$ *for* $t \in [0,T]$*, the previous line rewrites as*

$$X_t = x + \int_0^t X_s \hat{\pi}_s \mathrm{d}W_s^\theta, \quad 0 \le t \le T,$$

from which we see that X *is a non-negative local martingale, hence a supermartingale, under the measure* $\mathbb{P}^\theta$*. Moreover,* X *yields the representation*

$$X_t = x\mathcal{E}\big(\hat{\pi} \cdot W^\theta\big)_t, \quad 0 \le t \le T,$$

which implies $\mathbb{E}[\mathcal{E}\big(\hat{\pi} * W^\theta\big)_t$ *for every* $t \in [0,T]$*. Since* X *is strictly positive, we must have that* $\hat{\pi}$ *is locally square integrable under the measure* $\mathbb{P}^\theta$*. However, since* $\mathbb{P}^\theta$ *and* $\mathbb{P}$ *are equivalent, it follows that* $\hat{\pi}$ *is also square integrable under the measure* $\mathbb{P}$*.*

The following moment estimate is in the same spirit as Proposition 2.2.1 and forms an important stepping stone for the existence result in Theorem 2.3.1.

Lemma 2.3.1. *Let* $q \in \big(1, 1 + \frac{1}{2} \cdot \frac{1-p}{p}\big]$ *and let* $Z \in \mathbb{H}^2_{loc}(\mathbb{R}^{d_1})$*. Let* X *satisfy*

$$X_t = x + \int_0^t X_s\left(\frac{1}{1-p}\big(Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}\big)\right)\mathrm{d}W_s^{\mathcal{H}} + \int_0^t X_s\left(\frac{1}{1-p}\big(Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}\big)\theta_s^{\mathcal{H}}\right)\mathrm{d}s.$$

Then we have $X_T^{pq} \in L^1(\Omega, \mathcal{F}_T, \mathbb{P})$*.*

Proof. Recall that due to (2.4), we have

$$X_t^p = x^p\ \mathcal{E}\left(\frac{p}{1-p}(Z^{\mathcal{H}} + \theta^{\mathcal{H}}) * W^{\mathcal{H}}\right)_t \times \exp\left(\frac{1}{2}\int_0^t \frac{p}{1-p}\Big(|\theta_s^{\mathcal{H}}|^2 - |Z_s^{\mathcal{H}}|^2\Big)\mathrm{d}s\right).$$

This implies

$$
\begin{aligned}
X_t^{pq} = x^{pq}\ & \mathcal{E}\Big(q\frac{p}{1-p}\big(Z^{\mathcal{H}}+\theta^{\mathcal{H}}\big) * W^{\mathcal{H}}\Big)_t \\
& \times \exp\left(\frac{1}{2}\int_0^t \left|\frac{qp}{1-p}\big(Z_s^{\mathcal{H}}+\theta_s^{\mathcal{H}}\big)\right|^2 \mathrm{d}s - \frac{1}{2}\int_0^t q\Big|\frac{p}{1-p}\big(Z_s^{\mathcal{H}}+\theta_s^{\mathcal{H}}\big)\Big|^2 \mathrm{d}s \right.\\
& \quad \left. + \frac{1}{2}\int_0^t q\cdot\frac{p}{1-p}\big(|\theta_s^{\mathcal{H}}|^2 - |Z_s^{\mathcal{H}}|^2\big)\mathrm{d}s\right).
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
& \frac{1}{2}\int_0^t \left|\frac{qp}{1-p}\big(Z_s^{\mathcal{H}}+\theta_s^{\mathcal{H}}\big)\right|^2 \mathrm{d}s - \frac{1}{2}\int_0^t q\Big|\frac{p}{1-p}\big(Z_s^{\mathcal{H}}+\theta_s^{\mathcal{H}}\big)\Big|^2 \mathrm{d}s \\
& \quad + \frac{1}{2}\int_0^t q\cdot\frac{p}{1-p}\big(|\theta_s^{\mathcal{H}}|^2 - |Z_s^{\mathcal{H}}|^2\big)\mathrm{d}s \\
& = \frac{1}{2}q\cdot\frac{p}{1-p}\int_0^t \left((q-1)\cdot\frac{p}{1-p}|Z_s^{\mathcal{H}}+\theta_s^{\mathcal{H}}|^2 + |\theta_s^{\mathcal{H}}|^2 - |Z_s^{\mathcal{H}}|^2\right)\mathrm{d}s \\
& = \frac{1}{2}q\cdot\frac{p}{1-p}\int_0^t \left[\left((q-1)\cdot\frac{p}{1-p}-1\right)|Z_s^{\mathcal{H}}|^2 + \left((q-1)\cdot\frac{p}{1-p}+1\right)|\theta_s^{\mathcal{H}}|^2\right. \\
& \quad \left. + 2(q-1)\frac{p}{1-p}Z_s^{\mathcal{H}}\theta_s^{\mathcal{H}}\right]\mathrm{d}s \\
& \le \frac{1}{2}q\cdot\frac{p}{1-p}\int_0^t \left[\left(2(q-1)\frac{p}{1-p}-1\right)|Z_s^{\mathcal{H}}|^2 + \left(2(q-1)\frac{p}{1-p}+1\right)|\theta_s^{\mathcal{H}}|^2\right],
\end{aligned}
$$

where the last line follows from Young's inequality

$$
2(q-1)\frac{p}{1-p}z\cdot\theta \le (q-1)\frac{p}{1-p}\Big(|z|^2+|\theta|^2\Big), \quad z,\theta\in\mathbb{R}^{d_1}.
$$

Since $q \le 1 + \frac{1}{2} \cdot \frac{1-p}{p}$ it follows that $2(q-1)\frac{p}{1-p} - 1 \le 0$. This implies for every $0 \le t \le T$

$$\begin{aligned}
X_t^{pq} &= x^{pq}\ \mathcal{E}\Big(q\frac{p}{1-p}\big(Z^{\mathcal{H}} + \theta^{\mathcal{H}}\big) * W^{\mathcal{H}}\Big)_t \\
&\qquad \times \exp\left(\frac{1}{2}\int_0^t \left|\frac{qp}{1-p}\big(Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}\big)\right|^2 \mathrm{d}s - \frac{1}{2}\int_0^t q\Big|\frac{p}{1-p}\big(Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}\big)\Big|^2 \mathrm{d}s \right. \\
&\qquad \left. + \frac{1}{2}\int_0^t q \cdot \frac{p}{1-p}\big(|\theta_s^{\mathcal{H}}|^2 - |Z_s^{\mathcal{H}}|^2\big)\mathrm{d}s\right) \\
&\le x^{pq}\ \mathcal{E}\Big(q\frac{p}{1-p}\big(Z^{\mathcal{H}} + \theta^{\mathcal{H}}\big) * W^{\mathcal{H}}\Big)_t \\
&\qquad \times \exp\left\{\int_0^t \left(\frac{1}{2}\frac{qp}{1-p}\Big(2(q-1)\frac{p}{1-p} - 1\Big)|Z_s^{\mathcal{H}}|^2 + \Big(2(q-1)\frac{p}{1-p} + 1\Big)|\theta_s^{\mathcal{H}}|^2\right)\mathrm{d}s\right\} \\
&\le x^{pq}\mathcal{E}\Big(q\frac{p}{1-p}\big(Z^{\mathcal{H}} + \theta^{\mathcal{H}}\big) * W^{\mathcal{H}}\Big)_t \times \operatorname{esssup}\ \exp\left\{\int_0^T \Big(2(q-1)\frac{p}{1-p} + 1\Big)|\theta_s^{\mathcal{H}}|^2 \mathrm{d}s\right\} \\
&\quad \times \exp\left\{\int_0^t \frac{1}{2}q \cdot \frac{p}{1-p}\Big(2(q-1)\frac{p}{1-p} - 1\Big)|Z_s^{\mathcal{H}}|^2 \mathrm{d}s\right\} \\
&\le Cx^{pq}\mathcal{E}\Big(q\frac{p}{1-p}\big(Z^{\mathcal{H}} + \theta^{\mathcal{H}}\big) * W^{\mathcal{H}}\Big)_t,
\end{aligned} \tag{2.10}$$

where the last inequality follows by $2(q-1)\frac{p}{1-p} - 1 \le 0$ and

$$\operatorname{esssup}\ \exp\left\{\int_0^T \Big(2(q-1)\frac{p}{1-p} + 1\Big)|\theta_s^{\mathcal{H}}|^2 \mathrm{d}s\right\} \le C,$$

for some $C > 0$, because by **(H1)**, $\theta^{\mathcal{H}}$ is uniformly bounded. Now taking expectation in (2.10) and using the fact that $\mathcal{E}\Big(q\frac{p}{1-p}\big(Z + \theta^{\mathcal{H}}\big) * W^{\mathcal{H}}\Big)_t$ is a lower bounded local martingale, hence a supermartingale, we get

$$\mathbb{E}X_T^{qp} \le Cx^{qp}.$$

This concludes the proof. □

With the previous Lemma 2.3.1 at hand, we are in the position to prove a first important result which becomes instrumental for solving the FBSDE. Recall that our strategy to solve (X, Y, Z) is to find primal solutions of the optimization problems from (2.9). In the following result we show the existence of a terminal wealth which solves the optimization problem for the weighted marginal utility $w(x)$. The property that solving for $w(x)$ implies solving the primal problem $v(x)$ is the subject of Section 2.4.

Theorem 2.3.1. *There exists $\widehat{X}_T \in \mathcal{C}(x)$ such that*

$$w(x) = \mathbb{E}\left[\widehat{X}_T U'(\widehat{X}_T + H)\right].$$

Proof. Let $(X^n_T)_{n\in\mathbb{N}} \subset \mathcal{C}(x)$ be a sequence such that

$$\lim_{n\to\infty} \mathbb{E}\left[f(X^n_T)\right] = \lim_{n\to\infty} \mathbb{E}\left[X^n_T U'\big(X^n_T + H\big)\right] = w(x), \tag{2.11}$$

i.e. $(X^n_T)_{n\in\mathbb{N}}$ is a maximizing sequence attaining in the limit the optimal value $w(x) = \sup_{X_T\in\mathcal{C}(x)} \mathbb{E}\big[X_T U'\big(X_T + H\big)\big]$. Due to **(H2)** we have

$$0 \le X^n_T U'(X^n_T + H) = \frac{X^n_T}{(X^n_T + H)^{1-p}} \le |X^n_T|^p, \tag{2.12}$$

hence, according to Proposition 2.2.1, $X^n_T U'(X^n_T + H)$ is integrable. Now let us define martingales induced by the sequence X^n via

$$R^n_t := \mathbb{E}\left[X^n_T U'(X^n_T + H)|\mathcal{F}_t\right], \quad t \in [0,T], \tag{2.13}$$

and due to $0 \le X^n_T U'(X^n_T + H) \in L^1(\Omega, \mathcal{F}_T, \mathbb{P})$, all R^n's are true non-negative martingales. Invoking the predictable representation property of the Wiener filtration, there exist locally square integrable processes K^n, N^n with values in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d-d_1}$, respectively, such that

$$R^n_t = R^n_0 + \int_0^t K^n_s \mathrm{d}W^{\mathcal{H}}_s + \int_0^t N^n_s \mathrm{d}W^{\mathcal{O}}_s, \quad t \in [0,T]. \tag{2.14}$$

Since all R^n's are non-negative, we can rewrite their dynamics via

$$\mathrm{d}R^n_t = R^n_t \left(\frac{K^n_t}{R^n_t}\mathrm{d}W^{\mathcal{H}}_t + \frac{N^n_t}{R^n_t}\mathrm{d}W^{\mathcal{O}}_t\right), \quad R^n_T = X^n_T U'(X^n_T + H). \tag{2.15}$$

Now let us define

$$Z^n_t := (1-p)\frac{K^n_t}{R^n_t} - p\theta^{\mathcal{H}}_t \text{ and } \tilde{Z}^n_t := \frac{N^n_t}{R^n_t}. \tag{2.16}$$

Plugging (2.16) into (2.15), we see that $(R^n, Z^n, \tilde{Z}^n)$ solves

$$R^n_t = X^n_T U'(X^n_T + H) - \int_t^T R^n_s\big(\frac{1}{1-p}(Z^n_s + p\theta^{\mathcal{H}}_s)\big)\mathrm{d}W^{\mathcal{H}}_s - \int_t^T R^n_s \tilde{Z}^n_s \mathrm{d}W^{\mathcal{O}}_s, \quad t \in [0,T],$$

which is a BSDE of type (2.3). Since $X^n_T U'(X^n_T + H) \ge 0$, it follows from (2.12) that for every $q \ge 1$,

$$\big(X^n_T U'(X^n_T + H)\big)^q \le 1 + (X^n_T)^{pq}.$$

Applying Lemma 2.3.1, we obtain for some $q \in \big(1, 1 + \frac{1}{2}\,\frac{1-p}{p}\big]$ that $\sup_{n\in\mathbb{N}} \mathbb{E}|X^n_T|^{pq} \le Cx^{pq}$.

Hence, an application of Doob's maximal inequality yields

$$\mathbb{E}\sup_{0\le t\le T}|R^n_t|^q \le \left(\frac{q}{q-1}\right)^q \mathbb{E}|f(X^n_T)|^q \le \left(\frac{q}{q-1}\right)^q \left(1+\mathbb{E}|X^n_T|^{pq}\right)$$
$$\le \left(\frac{q}{q-1}\right)^q \left(1+Cx^{qp}\right). \tag{2.17}$$

This obviously implies $\sup_{n\in\mathbb{N}}\mathbb{E}|R^n_T|^q \le \left(\frac{q}{q-1}\right)^q\left(1+Cx^{qp}\right)$, so by the de la Vallée-Poussin criterion, the family of terminal values $R^n_T = X^n_T U'(X^n_T + H)$ is uniformly integrable. Since all R^n's are continuous, the BDG inequalities imply that this is equivalent to

$$\sup_{n\in\mathbb{N}}\mathbb{E}\langle R^n\rangle_T^{q/2} < \infty,$$

thus, $(R^n)_{n\in\mathbb{N}}$ is sequence of $\mathcal{H}^1$-bounded martingales[1]. Now according to Delbaen and Schachermayer [39, Theorem A], there exists a martingale R admitting the predictable representation

$$R_t = R_0 + \int_0^t K_s \mathrm{d}W^{\mathcal{H}}_s + \int_0^t N_s \mathrm{d}W^{\mathcal{O}}_s \tag{2.18}$$

with $K\in\mathbb{H}^2_{loc}(\mathbb{R}^{d_1})$, $N\in\mathbb{H}^2_{loc}(\mathbb{R}^{D-d_1})$ such that for subsequences

$$M^n \in \operatorname{conv}\{R^n, R^{n+1},\dots\}^2,$$

we have

$$\lim_{n\to\infty}\mathbb{E}\left(\int_0^T |M^n_s - R_s|^2\mathrm{d}s\right)^\gamma = 0, \quad \gamma\in(0,1). \tag{2.19}$$

Let us check that we have $\mathbb{E}[R_T] = w(x)$. Since all terminal values $R^n_T = X^n_T U'(X^n_T + H)$ are L^1-bounded, their convex combinations are also L^1-bounded. Now it follows from Komlos' Theorem (see e.g. Delbaen and Schachermayer [39, Theorem 1.4]) that $(R^n_T)_{n\in\mathbb{N}}$ also converges a.s. along convex combinations, i.e.

$$\lim_{n\to\infty}\sum_{m=n}^{N_n}\alpha^m X^m_T U'(X^m_T + H) = R_T,\ \mathbb{P}-a.s. \tag{2.20}$$

where for every $n\in\mathbb{N}$, we have $\sum_{m=n}^{N_n}\alpha^m = 1$. We recall that the sequence of terminal values $(R^n_T)_{n\in\mathbb{N}}$ is uniformly integrable. To see that convex combinations of $(R^n_T)_{n\in\mathbb{N}}$

[1]Here $\mathcal{H}^1$ denotes the space of all martingales M such that $\mathbb{E}\big[\langle M\rangle_T^{1/2}\big] < \infty$.
[2]$\operatorname{conv}(x_n, x_{n+1},\dots)$ denotes the set of all convex combinations of elements $x_n, x_{n+1},\dots$

inherit uniform integrability, let $q \in \left(1, 1 + \frac{1}{2}\,\frac{1-p}{p}\right]$. Jensen's inequality yields that

$$\sum_{m=n}^{N_n} \alpha^m R_T^m \leq \left(\sum_{m=n}^{N_n} \alpha^m |R_T^m|^q\right)^{\frac{1}{q}},$$

hence, using (2.17), we get

$$\begin{aligned}\mathbb{E}\left[\Big(\sum_{m=n}^{N_n} \alpha^m R_T^m\Big)^q\right] &\leq \mathbb{E}\left[\sum_{m=n}^{N_n} \alpha^m |R_T^m|^q\right] = \sum_{m=n}^{N_n} \alpha^m \mathbb{E}\left[|R_T^m|^q\right] \\ &\leq \left(\frac{q}{q-1}\right)^q \left(1 + Cx^{qp}\right) \cdot \sum_{m=n}^{N_n} \alpha^m = \left(\frac{q}{q-1}\right)^q \left(1 + Cx^{qp}\right) \\ &< \infty.\end{aligned}$$

By the de la Vallée-Poussin criterion, we see that convex combination of R_T^n are uniformly integrable. Combining the uniform integrability of convex combinations and the property (2.20), we get

$$\lim_{n\to\infty} \sum_{m=n}^{N_n} \mathbb{E}\left[\alpha^m X_T^m U'(X_T^m + H)\right] = w(x) = \mathbb{E}[R_T], \tag{2.21}$$

i.e. the optimal value $w(x)$ is also attained along convex combinations of the terminal values $(R_T^n)_{n\in\mathbb{N}}$.

We next make the observation that the mapping $f : (0,\infty) \times \Omega \to (0,\infty)$ defined as $f(x,\omega) = xU'(x + H(\omega)) = x(x + H(\omega))^{p-1}$ is $\mathcal{B}(\mathbb{R}_{>0}) \otimes \mathcal{F}_T$-measurable. Omitting the explicit dependence on ω, we have

$$f'(x) = (x + H)^{p-1}\Big(1 + (p-1)\frac{x}{x + H}\Big) > 0 \quad a.s. \tag{2.22}$$

hence, that the mapping f is strictly increasing for a.a. $\omega \in \Omega$. Let us now define

$$\widehat{X}_T := f^{-1}(R_T) \tag{2.23}$$

where $f^{-1} = f^{-1}(\cdot,\omega)$ is the inverse of f. Note that the range of f^{-1} is $(0,\infty)$, thus $\widehat{X}_T$ is a non-negative random variable. Hence, $\widehat{X}_T$ satisfies by construction $R_T = \widehat{X}_T U'\big(\widehat{X}_T + H\big)$, i.e. we also have

$$w(x) = \mathbb{E}\big[\widehat{X}_T U'\big(\widehat{X}_T + H\big)\big].$$

Let us now check that we have $\widehat{X}_T \in \mathcal{C}(x)$. To this end, let us associate a process to the terminal variable $\widehat{X}_T$ by using the optional decomposition result by El Karoui and Quenez [48] and Kramkov [81]. Denoting by $\mathcal{M}^e$ the set of all equivalent local martingale measures, there exists some $\pi \in \mathbb{H}^2_{loc}$ such that

$$\widehat{X}_t := \operatorname{esssup}_{Q\in\mathcal{M}^e} \mathbb{E}^Q\big[\widehat{X}_T \mid \mathcal{F}_t\big] = \widehat{X}_0 + \int_0^t \pi_s\big(\mathrm{d}W_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}\mathrm{d}s\big) - A_t, \quad t \in [0,T], \tag{2.24}$$

where A is a predictable and non-decreasing process satisfying $A_0 = 0$. Since $\widehat{X}_T$ is non-negative, the process X_t is also non-negative. By Theorem 7.2.2 from Pham [110], we have $\widehat{X}_0 = \sup_{Q\in\mathcal{M}^e} \mathbb{E}^Q[\widehat{X}_T] = x$.[3] Let us define

$$X_t := x + \int_0^t \pi_s\big(\mathrm{d}W_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}\mathrm{d}s\big),$$

which clearly satisfies $X_t \geq \widehat{X}_t$ a.s. for every $t \in [0,T]$. According to Remark 2.3.1, there exists $\widetilde{\pi} \in \mathbb{H}^2_{loc}$ such that $X_t = x + \int_0^t X_s\widetilde{\pi}_s\big(\mathrm{d}W_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}\mathrm{d}s\big)$, i.e. we have $X_T \in \mathcal{C}(x)$. Due to (2.22) the mapping $f(x) = x(x+H)^{p-1}$ is a.s. strictly increasing, hence we have

$$X_T U'(X_T + H) \geq \widehat{X}_T U'(\widehat{X}_T + H) \ a.s.$$

which implies $\mathbb{E}\big[X_T U'(X_T+H)\big] \geq \mathbb{E}\big[\widehat{X}_T U'(\widehat{X}_T + H)\big] = w(x)$. However, due to the optimality of $w(x)$, we must have $\mathbb{E}\big[X_T U'(X_T+H)\big] = \mathbb{E}\big[\widehat{X}_T U'(\widehat{X}_T+H)\big] = w(x)$. This yields $X_T = \widehat{X}_T$, meaning that $A \equiv 0$ and $\widehat{X} = X$. In particular, it follows that $\widehat{X}_T \in \mathcal{C}(x)$. This concludes the proof.

□

Constructing a solution to the FBSDE (2.1), (2.2) is now straightforwardly done by pocketing the outcomes of its proof. It proceeds by making use of the optimizing terminal wealth $X_T \in \mathcal{C}(x)$ and its integral representation

$$X_T = x + \int_0^T X_s\pi_s(\mathrm{d}W_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}\mathrm{d}s).$$

The process X serves as the forward component. Furthermore, a transformation of the control process π gives rise to another control process which eventually yields the value function of the backward equation. We can formulate the following result which uses Theorem 2.3.1 as a stepping stone.

Theorem 2.3.2. *There exists a triplet (X,Y,Z) which solves the FBSDE (2.1), (2.2). Moreover, X and Y are continuous processes and $Z = (Z^{\mathcal{H}}, Z^{\mathcal{O}}) \in \mathbb{H}^2_{loc}(\mathbb{R}^{d_1}) \times \mathbb{H}^2_{loc}(\mathbb{R}^{d-d_1})$*

Proof. By the proof of Theorem 2.3.1, there exists a random variable $X_T \in \mathcal{F}_T$ which has the representation $X_t = x + \int_0^t X_s\pi_s\big(\mathrm{d}W_s + \theta_s^{\mathcal{H}}\mathrm{d}s\big)$ with $\pi \in \mathbb{H}^2_{loc}$ such that $w(x) = \mathbb{E}\big[X_T U'(X_T+H)\big]$. Moreover, the random variable $X_T U'(X_T+H)$ is integrable and thus induces a martingale $R_t = \mathbb{E}\big[X_T U'(X_T+H)|\mathcal{F}_t\big]$. According to the martingale representation theorem there are processes $K \in \mathbb{H}^2_{loc}(\mathbb{R}^{d_1})$ and $N \in \mathbb{H}^2_{loc}(\mathbb{R}^{d-d_1})$ such that

$$R_t = X_T U'(X_T+H) - \int_t^T K_s \mathrm{d}W_s^{\mathcal{H}} - \int_t^T N_s \mathrm{d}W_s^{\mathcal{O}}, \quad t \in [0,T].$$

Now in analogy to the proof of Theorem 1.3.1, an application of Itô's formula to $Y_t =$

[3] Otherwise, we could consider $\tilde{X}_t := \widehat{X}_t + (x - \widehat{X}_0)$ which starts in $x > 0$.

$\log R_t - \log\big(X_t U'(X_t)\big)$ yields

$$
\begin{aligned}
Y_t = \log \frac{U'(X_T+H)}{U'(X_T)} &- \int_t^T \left[\frac{K_s}{R_s} - \frac{U''(X_s)}{U'(X_s)} X_s \pi_s - \pi_s\right] \mathrm{d}W_s^{\mathcal{H}} + \int_t^T \frac{N_s}{R_s} \mathrm{d}W_s^{\mathcal{O}} \\
&- \int_t^T \Bigg[-\frac{1}{2}\frac{|K_s|^2}{|R_s|^2} - \frac{1}{2}\frac{|N_s|^2}{|R_s|^2} - \left(\frac{U''(X_s)}{U'(X_s)} X_s \pi_s + \pi_s\right) \cdot \theta_s^{\mathcal{H}} \\
&\quad + \frac{|X_s \pi_s|^2}{2} \left(\left|\frac{U''(X_s)}{U'(X_s)}\right|^2 - \frac{U^{(3)}(X_s)}{U'(X_s)} \right) + \frac{|\pi_s|^2}{2} \Bigg] \mathrm{d}s.
\end{aligned}
$$

Setting

$$
Z_t^{\mathcal{H}} = \frac{K_t}{R_t} - \frac{\pi_t}{U'(X_t)}\big(X_t U''(X_t) + U'(X_t)\big) \text{ and } Z_t^{\mathcal{O}} = \frac{N_t}{R_t},
$$

we get

$$
\begin{aligned}
Y_t = \log \frac{U'(X_T+H)}{U'(X_T)} &- \int_t^T Z_s^{\mathcal{H}} \mathrm{d}W_s^{\mathcal{H}} - \int_t^T Z_s^{\mathcal{O}} \mathrm{d}W_s^{\mathcal{O}} - \int_t^T \Bigg[-\frac{1}{2} \frac{U^{(3)}(X_s)}{U'(X_s)} |X_s \pi_s|^2 \\
&- (Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}) \cdot \left(\frac{U''(X_s)}{U'(X_s)} X_s \pi_s + \pi_s\right) - \frac{U''(X_s)}{U'(X_s)} X_s |\pi_s|^2 - \frac{1}{2}|Z_s^{\mathcal{H}}|^2 - \frac{1}{2}|Z_s^{\mathcal{O}}|^2 \Bigg] \mathrm{d}s, \quad (2.25)
\end{aligned}
$$

which is a BSDE whose generator

$$
\begin{aligned}
f(s,\pi_s) = -\frac{1}{2}\frac{U^{(3)}(X_s)}{U'(X_s)} |X_s \pi_s|^2 &- (Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}) \cdot \left(\frac{U''(X_s)}{U'(X_s)} X_s \pi_s + \pi_s\right) \\
&- \frac{U''(X_s)}{U'(X_s)} X_s |\pi_s|^2 - \frac{1}{2}|Z_s^{\mathcal{H}}|^2 - \frac{1}{2}|Z_s^{\mathcal{O}}|^2
\end{aligned}
$$

depends on the control process π. By construction, we thus have $R_T = X_T U'(X_T) e^{Y_T} = X_T U'(X_T+H)$. The final step is to show that the optimal strategy has the representation

$$
\pi_t = -\frac{1}{1-p}\big(Z_t^{\mathcal{H}} + \theta_t^{\mathcal{H}}\big), \quad t \in [0,T].
$$

To do so, we need to identify the BSDE (Y,Z) as the transformed adjoint equation of a suitable Hamiltonian. More precisely, putting $p := e^Y$, Itô's formula yields

$$
\begin{aligned}
\mathrm{d}p_t = p_t &\Big(Z_t^{\mathcal{H}} \mathrm{d}W_t^{\mathcal{H}} + Z_t^{\mathcal{O}} \mathrm{d}W_t^{\mathcal{O}} \Big) \\
&+ p_t \Big[-\frac{1}{2} \; \frac{U^{(3)}(X_t)}{U'(X_t)} |X_t \pi_t|^2 - \Big(\frac{U''(X_t) X_t}{U'(X_t)} + 1 \Big) \pi_t \theta_t^{\mathcal{H}} - \frac{U''(X_t)}{U'(X_t)} X_t |\pi_t|^2 \\
&\quad - \Big(\frac{U''(X_t)}{U'(X_t)} X_t + 1 \Big) Z_t^{\mathcal{H}} \pi_t \Big] \mathrm{d}t,
\end{aligned}
$$

which by $q = (q^{\mathcal{H}}, q^{\mathcal{O}}) := e^Y Z = e^Y (Z^{\mathcal{H}}, Z^{\mathcal{O}})$ becomes

$$\begin{aligned} \mathrm{d}p_t &= q_t^{\mathcal{H}} \mathrm{d}W_t^{\mathcal{H}} + q_t^{\mathcal{O}} \mathrm{d}W_t^{\mathcal{O}} \\ &\quad - \Bigg[p_t \cdot \Big(-\frac{1}{2}\, \frac{U^{(3)}(X_t)}{U'(X_t)} |X_t \pi_t|^2 - \Big(\frac{U''(X_t) X_t}{U'(X_t)} + 1 \Big) \pi_t \theta_t^{\mathcal{H}} - \frac{U''(X_t)}{U'(X_t)} X_t |\pi_t|^2 \Big) \\ &\qquad + q_t^{\mathcal{H}} \cdot \Big(\frac{U''(X_t)}{U'(X_t)} X_t + 1 \Big) \pi_t \Bigg] \mathrm{d}t. \end{aligned}$$

The pair (p, q) is indeed a BSDE because we have the terminal condition $p_T = \frac{U'(X_T + H)}{U'(X_T)}$. In order to characterize the optimal strategy π, we invoke the maximum principle from Peng [107], see also Section 1.4.1. Introducing $\tilde{X} := U(X)$, we obtain in the same fashion as in (1.31) that the Hamiltonian is given by

$$\begin{aligned} H(t, x, \pi, p, q) = p[&U'(U^{-1}(x)) U^{-1}(x) \pi \theta_t^{\mathcal{H}} + \frac{1}{2} U''(U^{-1}(x) U^{-1}(x)^2 |\pi|^2] \\ &+ q U'(U^{-1}(x)) U^{-1}(x) \pi. \end{aligned}$$

Evaluated along the optimal path and taking into account that $X = U^{-1}(\tilde{X})$, we have

$$H(t, X_t, \pi_t, p_t, q_t) = p_t \cdot \left(U'(X_t) X_t \pi_t \theta_t^{\mathcal{H}} + \frac{1}{2}\, U''(X_t) X_t^2 |\pi_t|^2 \right) + q_t \cdot \left(U'(X_t) X_t \pi_t \right). \tag{2.26}$$

Moreover, it is straightforward to check that we have

$$\mathrm{d}p_t = -\frac{\partial}{\partial x} H(t, U^{-1}(\tilde{X}_t), \pi_t, p_t, q_t) \mathrm{d}t + q_t^{\mathcal{H}} \mathrm{d}W_t^{\mathcal{H}} + q_t^{\mathcal{O}} \mathrm{d}W_t^{\mathcal{O}},$$

with the terminal condition $p_T = \frac{U'(X_T + H)}{U'(X_T)}$. Now by the maximum principle, p is the adjoint equation associated to the Hamiltonian (2.26) and we have

$$H(t, X_t, \pi_t, p_t, q_t) = \max_{\pi \in \mathbb{R}^{d_1}} H(t, X_t, \pi, p_t, q_t),$$

because π is the optimal control. In order to find a representation for this optimal control, we solve for $\frac{\partial}{\partial \pi} H = 0$ and find

$$\pi_t = -\frac{U'(X_t)}{U''(X_t) X_t} \Big(\frac{q_t^{\mathcal{H}}}{p_t} + \theta_t^{\mathcal{H}} \Big) = -\frac{1}{1-p} \big(Z_t^{\mathcal{H}} + \theta_t^{\mathcal{H}} \big). \tag{2.27}$$

This means that whenever a control is optimal, it must coincide with $-\frac{1}{1-p}\big(Z_t^{\mathcal{H}} + \theta_t^{\mathcal{H}}\big)$. By plugging (2.27) into (2.25), it becomes straightforward to see that (Y, Z) satisfies the backward equation from (2.2). We obviously have that X and Y are continuous, and by construction, $Z^{\mathcal{H}}$ and $Z^{\mathcal{O}}$ are locally square integrable. □

It is worth to mention that so far, all arguments that we have employed do not rely on the specific assumption that the utility function is of power type. In fact, once we make the general assumption that U has bounded polynomial growth with

(H) $|U(x)| \leq C\big(1 + x^{\gamma}\big), \quad x > 0,$

for some $\gamma \in (0,1)$, then (up to technical details) all results still remain true. We anticipate that the techniques from Theorem 2.3.1 and Theorem 2.3.2 can be straightforwardly generalized to prove an existence result for the coupled FBSDE (1.26) for a general utility function U and its associated FBSDE

$$\begin{cases} X_t = x - \int_0^t \frac{U'(X_s)}{U''(X_s)}(Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}})\mathrm{d}W_s^{\mathcal{H}} - \int_0^t \frac{U'(X_s)}{U''(X_s)}(Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}})\theta_s \mathrm{d}s, \quad x > 0, \\ Y_t = Y_T - \int_t^T \left[(|Z_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}|^2) \left(1 - \frac{1}{2} \frac{U^{(3)}(X_s)U'(X_s)}{|U''(X_s)|^2} \right) - \frac{1}{2}|Z_s|^2 \right] \mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \end{cases}$$

where $Y_T = \log\left(\frac{U'(X_T+H)}{U'(X_T)}\right)$.

2.4 First order optimality and uniqueness

In this section, we briefly investigate the role played by the quantity $X_T U'(X_T + H)$. It turns out that once this quantity is maximized with respect to X_T, we get a sufficient criterion for X_T to be optimal for the primal problem of optimizing expected terminal utility $U(X_T + H)$. In a first step, we check that if X_T solves $w(x) = \sup_{X_T \in \mathcal{C}(x)} \mathbb{E}\big[X_T U'\big(X_T + H\big)\big]$, then X_T also solves $\sup_{X_T \in \mathcal{C}(x)} \mathbb{E}\left[U\big(X_T + H\big)\right]$, i.e. optimizing $X_T U'\big(X_T + H\big)$ is sufficient for optimizing $U\big(X_T + H\big)$, see Lemma 2.4.1. In a second step, we identify the quantity $U'(X_T + H)$ as the density of an equivalent local martingale measure which acts on the terminal wealth X_T as a drift tilting factor. This approach of seeking for equivalent martingale measures is the central theme of the martingale dual approach pioneered by Pliska [112], Karatzas et al. [73] and extended to the general semimartingale framework by Kramkov and Schachermayer [80].

Lemma 2.4.1. *Let $x \in (0,\infty)$ be such that $v(x) < \infty$. Assume that $X_T \in \mathcal{C}(x)$ is such that $w(x) = \mathbb{E}\left[X_T U'\big(X_T + H\big)\right] < \infty$ holds. Then we have*

$$v(x) = \mathbb{E}\left[U\big(X_T + H\big)\right].$$

Moreover, X_T is unique.

Proof. Let (X,Y,Z) be the solution to the FBSDE (2.1), (2.2) whose existence is guaranteed by Theorem 2.3.2. Now suppose that there exists $\widetilde{X}_T \in \mathcal{C}(x)$ such that

$$\mathbb{E}\left[U\big(X_T + H\big)\right] < \mathbb{E}\left[U\big(\widetilde{X}_T + H\big)\right].$$

Setting $D = U'(X)e^Y$, a straightforward application of Itô's formula gives

$$D_t = U'(x)e^{Y_0}\mathcal{E}\left(\int -\theta^{\mathcal{H}}\mathrm{d}W^{\mathcal{H}} + \int Z^{\mathcal{O}}\mathrm{d}W^{\mathcal{O}}\right)_t,$$
$$D_t\hat{X}_t = xD_0\mathcal{E}\left(\int \left(\hat{\pi} - \theta^{\mathcal{H}}\right)\mathrm{d}W^{\mathcal{H}} + \int Z^{\mathcal{O}}\mathrm{d}W^{\mathcal{O}}\right)_t,$$

for every $\hat{X}_t = x + \int_0^t \hat{X}_s\hat{\pi}_s\left(\mathrm{d}W_s^{\mathcal{H}} + \theta_s^{\mathcal{H}}\mathrm{d}s\right)$, i.e. $D_t\hat{X}_t$ is a non-negative local martingale, hence a supermartingale. It is thus clear that we have $\mathbb{E}\left[D_T\hat{X}_T\right] \le xD_0$. Note also that $D_TX_T = X_TU'\left(X_T + H\right)$, hence $w(x) = \mathbb{E}\left[D_TX_T\right] = xD_0$ which implies that D_tX_t is a martingale. Now using the concavity of U we get

$$\begin{aligned}
0 &< \mathbb{E}\left[U\left(\widetilde{X}_T + H\right) - U\left(X_T + H\right)\right] \le \mathbb{E}\left[U'\left(X_T + H\right)\left(\widetilde{X}_T - X_T\right)\right]\\
&= \mathbb{E}\left[D_T\widetilde{X}_T - D_T\widehat{X}_T\right]\\
&= \mathbb{E}\left[D_T\widetilde{X}_T\right] - w(x)\\
&\le xD_0 - xD_0\\
&= 0.
\end{aligned}$$

By $0 < 0$ we have a contradiction and it therefore follows that X_T must be also optimal for the problem $\sup_{X_T\in\mathcal{C}(x)}\mathbb{E}\left[U\left(X_T + H\right)\right]$, i.e. we must have $v(x) = \mathbb{E}\left[U\left(X_T + H\right)\right]$.

Now let us show uniqueness. Suppose that there exists another $\overline{X}_T \in \mathcal{C}(x)$ such that $\mathbb{E}\left[U(\overline{X}_T + H)\right] = \mathbb{E}\left[U(X_T + H)\right]$. Using again the concavity of U and the optimality of $w(x)$, we get

$$\begin{aligned}
0 &= \mathbb{E}\left[U(\overline{X}_T + H)\right] - \mathbb{E}\left[U(X_T + H)\right]\\
&= \mathbb{E}\left[U'(X_T + H)(\overline{X}_T - X_T)\right]\\
&\le 0,
\end{aligned}$$

This can only hold true if we have $\overline{X}_T = X_T$. □

In the previous proof, the key point is the strict concavity of U which results in the inequality

$$\mathbb{E}\left[U\left(\widetilde{X}_T + H\right) - U\left(X_T + H\right)\right] \le \mathbb{E}\left[U'\left(X_T + H\right)\left(\widetilde{X}_T - X_T\right)\right].$$

This inequality highlights the importance of the quantity $X_TU'\left(X_T + H\right)$ which is the backbone of the dual martingale approach from Karatzas et al. [73]. More precisely, the dual approach interprets the optimization of the quantity $X_TU'\left(X_T + H\right) = X_TU'(X_T)e^{Y_T}$ as finding the "best" expectation of X_T under a suitable martingale measure. The problem $\sup_{X_T\in\mathcal{C}(x)}\mathbb{E}\left[U\left(X_T + H\right)\right]$ is recast into the martingale measure dual formulation $\sup_{\mathbb{Q}\in\mathcal{M}^e}\mathbb{E}^{\mathbb{Q}}\left[X_T\right]$ where $\mathcal{M}^e$ denotes the set of all local martingale measures. This dual problem transfers the primal problem to the level of a first order optimality criterion. Once the optimal (density of the) measure is found, one can find the optimal wealth process X and the associated optimal strategy π. This is the gist of the martingale dual approach.

In our approach from Theorem 2.3.2, we can attach another meaning to the BSDE (Y, Z). In fact, we identify e^Y as the adjoint equation for the Hamiltonian system of the utility maximization problem. Thus, rather than finding a suitable dual formulation, (Y, Z) appears as a natural consequence of the maximum principle, a well-known approach to solve stochastic control problems.

3 BSDEs related to BSPDEs and applications to utility maximization

The aim of this chapter to study BSDEs arising from a particular class of backward stochastic partial differential equations (BSPDEs) that is intimately related to utility maximization problems with respect to general utility functions. This is in contrast to Chapter 1 and Chapter 2 where utility maximization problems were studied by means of non-linear fully coupled FBSDEs. We provide existence and uniqueness results for this particular class of BSPDEs under consideration by reducing them to ordinary BSDEs. In contrast to Chapter 1 and Chapter 2, the approach here allows to outline a numerical recipe. We then study utility maximization problems on incomplete markets whose dynamics is governed by continuous semimartingales. Adapting standard methods that solve the utility maximization problem using BSDEs, we give solutions to the portfolio optimization problem which involve the delivery of a liability at maturity. We illustrate our study by numerical simulations for selected examples.

3.1 BSPDEs and their reduction to BSDEs

In this section we study the particular class of BSPDEs derived in Mania and Tevzadze [92] and Musiela and Zariphopoulou [97]. Our focus is on the link to BSDEs. As already discussed in Mania and Tevzadze [92], in the context of utility optimization problems, particular BSDEs give rise to solutions to BSPDEs once the utility function is classical. We come back to this topic later on in this chapter. Let us first start with a depiction of the probabilistic setup we are working with.

3.1.1 Preliminaries and notations

Let $T \in \mathbb{R}_+ = [0,\infty)$ denote the terminal time. We work on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ endowed with a continuous and complete filtration $(\mathcal{F}_t)_{0\le t\le T}$, which governs an $\mathbb{R}^{d\times 1}$-valued continuous local martingale $M = (M^1,\dots,M^d)^{\mathsf{tr}}$. By tr we denote the transpose of real valued vectors. We call a filtration $(\mathcal{F}_t)$ continuous if every $\mathbb{R}$-valued square integrable, $(\mathcal{F}_t)$-adapted martingale N is continuous and yields the representation

$$N_t = N_0 + \int_0^t Z_s^{\mathsf{tr}} \mathrm{d}M_s + L_t,\ 0 \le t \le T, \tag{3.1}$$

where Z is a predictable process with values in $\mathbb{R}^{d\times 1}$ and L a real valued square integrable martingale that is strongly orthogonal to M, i.e. $\langle M^k, L\rangle = 0$ for every $k \in \{1,\dots,d\}$. Given a probability measure $\mathbb{Q}$ on $(\Omega,\mathcal{F})$, we denote by $\mathbb{E}^{\mathbb{Q}}$ the expectation with respect to $\mathbb{Q}$ and omit the superscript if $\mathbb{Q}$ is equal to the original measure $\mathbb{P}$. The Kunita-Watanabe inequality (see e.g. Theorem 25 in Protter [114]) implies that every covariation process $\langle M^k, M^l\rangle$, $k,l \in \{1,\dots,d\}$, is absolutely continuous with respect to the process $C = \sum_{k=1}^d \langle M^k, M^k\rangle$. Hence there exists an increasing and bounded continuous process K, e.g. given by $K = \arctan(C)$, such that the quadratic variation $\langle M, M\rangle$ satisfies the structural representation

$$\mathrm{d}\langle M, M\rangle = \sigma\sigma^{\mathsf{tr}}\mathrm{d}K, \tag{3.2}$$

where σ is a predictable process with values in $\mathbb{R}^{d\times d}$ such that $\sigma_t\sigma_t^{\mathsf{tr}}$ is almost surely invertible for every $t \in [0,T]$. We denote the Euclidean norm of vectors $x \in \mathbb{R}^{d\times 1}$ by $|x| = (x^{\mathsf{tr}}x)^{\frac{1}{2}}$. Moreover for $m \in \mathbb{N}$ we denote throughout this chapter

- $\mathcal{H}^2(\mathbb{R}^m, \mathbb{Q}, \sigma)$ as the space of all predictable processes Z taking values in $\mathbb{R}^{m\times 1}$ such that $\mathbb{E}^{\mathbb{Q}}\left[\int_0^T |\sigma_s Z_s|^2 \mathrm{d}K_s\right] < \infty$;
- $\mathcal{S}^\infty(\mathbb{R}^m)$ as the space of all bounded continuous $\mathbb{R}^{m\times 1}$-valued processes $(Y_t)_{0\le t\le T}$;
- $\mathcal{M}^2([0,T], \mathbb{Q})$ as the space of all real-valued and square integrable martingales under the measure $\mathbb{Q}$, adapted to $(\mathcal{F}_t)_{0\le t\le T}$ and starting in zero.

If there is no ambiguity about m, $\mathbb{Q}$ or σ we omit referencing to $\mathbb{R}^m$, $\mathbb{Q}$ and σ and simply write $\mathcal{H}^2$, $\mathcal{S}^\infty$ and $\mathcal{M}^2$.

3.1.2 BSPDEs related to utility maximization problems

For utility maximization problems, it is well-known that for the standard utility functions of logarithmic, exponential and power type, linear and quadratic BSDEs provide a unique solution for the optimization problem. A prominent approach of deriving from the problem

$$\sup_\pi \mathbb{E}[U(x + \int_0^T \pi_u \mathrm{d}S_u + H)] \tag{3.3}$$

a BSDE is assuming certain regularity conditions that permit the application of Itô's formula and taking advantage of the explicit form of the standard utility functions. For general smooth deterministic utility functions $U : \mathbb{R}_+ \to \mathbb{R}$ (i.e. U is continuously differentiable, strictly increasing and strictly concave) satisfying the Inada conditions

$$U'(0) = \lim_{x\to 0} U'(x) = \infty,$$
$$U'(\infty) = \lim_{x\to\infty} U'(x) = 0,$$

there is an alternative approach based on the generalized Itô's formula, also known as *Itô-Wentzell's formula*. Making use of this, it has been shown in Mania and Tevzadze [92] and Musiela and Zariphopoulou [97] that solving the optimization problem (3.3) with a general utility function U leads to the particular equation

$$V(t,x) = U(x) - \frac{1}{2}\int_t^T \frac{(\varphi_x(s,x) + \lambda_s V_x(s,x))^{\mathrm{tr}}}{V_{xx}(s,x)} \mathrm{d}\langle M, M\rangle_s (\varphi_x(s,x) + \lambda_s V_x(s,x)) \\ - \int_t^T \varphi(s,x)^{\mathrm{tr}} \mathrm{d}M_s - \int_t^T \mathrm{d}L(s,x), \quad 0 \le t \le T, \tag{3.4}$$

where λ is a given $\mathbb{R}^{d\times 1}$-dimensional predictable process and $L(\cdot, x)$ is a continuous one-dimensional martingale strongly orthogonal to M for all x. Since this equation is characterized by its terminal condition $V(T,x) = U(x)$ and involves the partial derivatives φ_x, V_x and V_{xx}, it is called a *backward stochastic partial differential equation* (BSPDE). It is clear that the non-linearity of the driver of equation (3.4) does not fall into the class of BSPDEs studied in Hu and Peng [59], Peng [105] and Hu et al. [61]. However, if one has a solution (V,φ) to (3.4), one can characterize the optimal trading strategy and the optimal wealth process in terms of the BSPDE (we shall elaborate on this in more details in Section 3.2). We stress here that existence (and uniqueness) of solutions to equation (3.4) have neither been shown in Mania and Tevzadze [92] nor in Musiela and Zariphopoulou [97] for the case of a general utility function U. As emphasized by the authors of these works, the BSPDE (3.4) yields a verification tool, but solving it directly remains a challenging task.

We shall get across the message here that we are also not solving the BSPDE (3.4). We will rather provide a discussion of an *ordinary* version of (3.4) which on the one hand does not belong to the standard class of Lipschitz or quadratic growth BSDEs and which on the other hand provides an alternative BSDE interpretation of solutions to utility maximization with respect to the power and the exponential function. More precisely, we investigate the BSDE

$$V_t = \xi + \alpha \int_t^T \frac{(\varphi_s + \lambda_s V_s)^{\mathrm{tr}}}{V_s} \mathrm{d}\langle M, M\rangle_s (\varphi_s + \lambda_s V_s) - \int_t^T \varphi_s^{\mathrm{tr}} \mathrm{d}M_s - (L_T - L_t), \tag{3.5}$$

for which we assume that $\xi \in L^\infty(\mathbb{R})$ is an $\mathcal{F}_T$-measurable random variable that is bounded away from zero, i.e. there exists a constant $c > 0$ such that $\xi \ge c > 0$ holds $\mathbb{P}$-almost surely. We say that (V,φ,L) is a *solution* of the BSDE (3.5) if

$$(V,\varphi,L) \in \mathcal{S}^\infty(\mathbb{R}) \times \mathcal{H}^2\big(\mathbb{R}^d, \mathbb{P}, \sigma\big) \times \mathcal{M}^2([0,T]) \text{ such that } V_t > 0 \text{ for all } t \in [0,T].$$

The positivity condition on V stems from the fact that V will play the role of the value function. In the following we denote the solution spaces without their dimensions and parameters. In Section 4 of Mania and Tevzadze [92], particular cases in which this BSDE admits a solution are considered, with proof methods related to particular choices of ξ and the constant α. Yet, we shall provide existence and uniqueness results for (3.5) using a different method. Key to showing existence and uniqueness of (3.5) is to find a suitable transformation relating (3.5) to the quadratic BSDE

$$Y_t = B + \int_t^T f(s, Z_s)\mathrm{d}K_s - \int_t^T Z_s^{\mathrm{tr}}\mathrm{d}M_s - (N_T - N_t) + \frac{1}{2}\int_t^T \mathrm{d}\langle N, N\rangle_s, \tag{3.6}$$

$$f(t,z) = \Big(\alpha + \frac{1}{2}\Big)|z^{\mathrm{tr}}\sigma_t|^2 + \alpha\left(z^{\mathrm{tr}}\sigma_t\sigma_t^{\mathrm{tr}}\lambda_t + \lambda_t^{\mathrm{tr}}\sigma_t\sigma_t^{\mathrm{tr}}z\right) + \alpha|\sigma_t^{\mathrm{tr}}\lambda_t|^2, \tag{3.7}$$

where B is a real valued bounded $\mathcal{F}_T$-measurable random variable, α a real number and λ an $\mathbb{R}^{d\times 1}$-valued predictable process. For this purpose we will use a *logarithmic coordinate change* which disentangles the denominator term of the driver of (3.5) and transforms it into the driver of a quadratic BSDE of the form (3.6). This type of transformation is employed in Hyndman [64] in a Brownian motion setting to prove existence and uniqueness of a fully-coupled FBSDE with a quadratic growth backward equation by solving an equivalent linear FBSDE. We also remark that this coordinate change can be found in the work of Kobylanski [77] and Morlais [96]. We proceed similarly in our setup of a general continuous stochastic basis. Note that existence and uniqueness of the BSDE (3.6) have been studied in Morlais [96] and Tevzadze [123] and some representations of the solution have been derived in Imkeller et al. [68]. The next result summarizes the conditions that guarantee existence and uniqueness of (3.6).

Lemma 3.1.1. *Let B be an $\mathcal{F}_T$-measurable random variable which is bounded. Assume that there exists a constant $C > 0$ such that we have $\mathbb{P}$-almost surely $\int_0^T |\sigma_s^{\mathrm{tr}}\lambda_s|^2\mathrm{d}K_s \le C$. Then, there exists a unique triplet $(Y, Z, N) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{M}^2([0,T])$ which solves the BSDE* (3.6).

Proof. The hypothesis that $\int_0^T |\sigma_s^{\mathrm{tr}}\lambda_s|^2\mathrm{d}K_s \le C$ implies that the process $|\sigma^{\mathrm{tr}}\lambda|$ is almost surely bounded. Hence the driver of equation (3.6) satisfies almost surely

$$\begin{aligned}|f(t,z)| &\le |\alpha + \frac{1}{2}|\ |\sigma_t^{\mathrm{tr}}z|^2 + 2|\alpha\ \sigma_t^{\mathrm{tr}}\lambda_t|\ |\sigma_t^{\mathrm{tr}}z| + |\alpha|\ |\sigma_t^{\mathrm{tr}}\lambda_t|^2 \\ &\le \gamma\big(|\alpha|\ |\sigma_t^{\mathrm{tr}}\lambda_t|^2 + |\sigma_t^{\mathrm{tr}}z|^2\big), \quad t \in [0,T],\ z \in \mathbb{R}^{d\times 1},\end{aligned}$$

where γ is some non-negative real constant. Then, by Proposition 2 from Tevzadze [123] or Theorems 2.5 and 2.6 from Morlais [96], there exists a unique triplet $(Y, Z, N) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{M}^2([0,T])$ which solves BSDE (3.6). □

The following result shows that depending on the sign of the constant α, the process V either becomes a sub- or a supermartingale. In the case of a supermartingale, V is even bounded away from zero.

Lemma 3.1.2. *Let $(V, \varphi, L) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{M}^2$ be a solution of*

$$V_t = \xi + \alpha\int_t^T \frac{(\varphi_s + \lambda_s V_s)^{\mathrm{tr}}}{V_s}\mathrm{d}\langle M, M\rangle_s\,(\varphi_s + \lambda_s V_s) - \int_t^T \varphi_s^{\mathrm{tr}}\mathrm{d}M_s - (L_T - L_t),$$

where the terminal condition ξ is a non-negative and bounded $\mathcal{F}_T$-measurable random variable such that there exists a constant $c > 0$ for which $\xi \ge c$ holds $\mathbb{P}$-almost surely.

Then, if $\alpha > 0$, V is a supermartingale which bounded from below by $c > 0$ and if $\alpha < 0$, V is a submartingale.

Proof. Let $\alpha > 0$. By the definition of a solution (V, φ, L) of the BSDE, we have that $V_t > 0$ for every $t \in [0, T]$. Now we have for $0 \le s \le t \le T$

$$
\begin{aligned}
V_s &= \mathbb{E}\Big[\xi + \alpha \int_s^T \frac{(\varphi_u + \lambda_u V_u)^{\mathsf{tr}}}{V_u} \mathrm{d}\langle M, M\rangle_u \,(\varphi_u + \lambda_u V_u)\,\Big|\mathcal{F}_s\Big] \\
&\ge \mathbb{E}\Big[\xi + \alpha \int_t^T \frac{(\varphi_u + \lambda_u V_u)^{\mathsf{tr}}}{V_u} \mathrm{d}\langle M, M\rangle_u \,(\varphi_u + \lambda_u V_u)\,\Big|\mathcal{F}_s\Big] \\
&= \mathbb{E}\Big[V_t\Big|\mathcal{F}_s\Big],
\end{aligned}
$$

showing that V is a supermartingale. This implies

$$
V_t \ge \mathbb{E}\Big[V_T\Big|\mathcal{F}_t\Big] = \mathbb{E}\Big[\xi\Big|\mathcal{F}_t\Big] \ge \mathbb{E}\Big[c\Big|\mathcal{F}_t\Big] = c > 0
$$

for every $t \in [0, T]$, i.e. V is even bounded away from zero. If $\alpha < 0$, the same arguments show that V is a submartingale. $\square$

We can now prove the existence and uniqueness of solutions to (3.5).

Proposition 3.1.1. *Let $\xi \in L^\infty$ be a non-negative $\mathcal{F}_T$-measurable random variable which is bounded away from zero. Let λ be a predictable process such that almost surely*

$$
\int_0^T |\sigma_s^{\mathsf{tr}} \lambda_s|^2 \mathrm{d}K_s \le C
$$

for some real constant $C > 0$. Let α be a non-zero real constant. Then, the BSDE

$$
V_t = \xi + \alpha \int_t^T \frac{(\varphi_s + \lambda_s V_s)^{\mathsf{tr}}}{V_s} \mathrm{d}\langle M, M\rangle_s \,(\varphi_s + \lambda_s V_s) - \int_t^T \varphi_s^{\mathsf{tr}} \mathrm{d}M_s - (L_T - L_t)
$$

admits a unique solution $(V, \varphi, L) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{M}^2$.

Proof. By the assumptions on the terminal variable ξ, the random variable $B := \log(\xi)$ is well-defined and also belongs to $L^\infty(\mathbb{R})$. According to Lemma 3.1.1, there exists a unique triplet $(Y, Z, N) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{M}^2$ which satisfies BSDE (3.6) with the terminal condition $B = \log(\xi)$. Let us set for $t \in [0, T]$

$$
P_t := e^{Y_t}, \tag{3.8}
$$

$$
Q_t := e^{Y_t} Z_t = P_t Z_t, \tag{3.9}
$$

$$
R_t := \int_0^t e^{Y_s} \mathrm{d}N_s = \int_0^t P_s \mathrm{d}N_s, \tag{3.10}
$$

which by the existence of (Y, Z, N) are well-defined. Note that since Y is bounded, the range of the process P lies in a compact subset of the real line that is bounded away from zero. This implies that $Q \in \mathcal{H}^2$ and hence R becomes a square integrable martingale which is, due to the orthogonality of N to M, also orthogonal to M. An application of Itô's formula in conjunction with (3.2) and (3.6) yields

$$\begin{aligned}
P_t &= \xi + \alpha \int_t^T \left(\frac{|\sigma_s^{\mathrm{tr}} Q_s|^2}{P_s} + \frac{Q_s^{\mathrm{tr}} \sigma_s \sigma_s^{\mathrm{tr}} \lambda_s P_s}{P_s} + \frac{P_s \lambda_s^{\mathrm{tr}} \sigma_s \sigma_s^{\mathrm{tr}} Q_s}{P_s} + |\sigma_s^{\mathrm{tr}} \lambda_s|^2 P_s \right) \mathrm{d}K_s \\
&\quad - \int_t^T Q_s^{\mathrm{tr}} \mathrm{d}M_s - (R_T - R_t) \\
&= \xi + \alpha \int_t^T \frac{(Q_s + \lambda_s P_s)^{\mathrm{tr}}}{P_s} \mathrm{d}\langle M, M\rangle_s \, (Q_s + \lambda_s P_s) - \int_t^T Q_s^{\mathrm{tr}} \mathrm{d}M_s - (R_T - R_t).
\end{aligned}$$

But this is exactly BSDE (3.5), hence setting $(V, \varphi, L) = (P, Q, R)$ we have found a solution in $\mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{M}^2$.

In order to prove uniqueness, assume that (V^1, φ^1, L^1) and (V^2, φ^2, L^2) are two solutions of (3.5). We apply the *logarithmic change* which in a first step amounts to defining a triplet $(Y^i, Z^i, N^i) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{M}^2$ via

$$Y_t^i = \log(V_t^i), \quad Z_t^i = \frac{\varphi_t^i}{Y_t^i}, \quad N_t^i = \int_0^t \frac{1}{V_s^i} \mathrm{d}L_s^i, \quad t \in [0, T].$$

Note that by the definition of solutions $(V^i, \varphi^i, L^i), i = 1, 2$, we have $V_t^i > 0$ for every $t \in [0, T]$, hence Y^i is well defined. In a next step we apply Itô's formula to Y^i and get the quadratic BSDE

$$\begin{aligned}
Y_t^i &= B + \int_t^T \alpha (Z_s^i + \lambda_s)^{\mathrm{tr}} \mathrm{d}\langle M, M\rangle_s \big(Z_s^i + \lambda_s\big) + \frac{1}{2} \int_t^T |Z_s^i|^2 \mathrm{d}\langle M, M\rangle_s \\
&\quad - \int_t^T (Z_s^i)^{\mathrm{tr}} \mathrm{d}M_s - (N_T^i - N_t^i) + \frac{1}{2}\big(\langle N^i, N^i\rangle_T - \langle N^i, N^i\rangle_t\big) \\
&= B + \int_t^T \Big\{ \big(\alpha + \frac{1}{2}\big) |\sigma_s^{\mathrm{tr}} Z_s^i|^2 + \alpha\big((Z_s^i)^{\mathrm{tr}} \sigma_s \sigma_s^{\mathrm{tr}} \lambda_s + \lambda_s^{\mathrm{tr}} \sigma_s \sigma_s^{\mathrm{tr}} Z_s^i\big) + \alpha |\sigma_s^{\mathrm{tr}} \lambda_s|^2 \Big\} \mathrm{d}K_s \\
&\quad - \int_t^T (Z_s^i)^{\mathrm{tr}} \mathrm{d}M_s - (N_T^i - N_t^i) + \frac{1}{2}\big(\langle N^i, N^i\rangle_T - \langle N^i, N^i\rangle_t\big).
\end{aligned} \tag{3.11}$$

For BSDEs of this type, comparison principles are available, see e.g. Theorem 2 in Tevzadze [123] or Theorem 2.7 in Morlais [96]. Taking into consideration that we have $Y_T^1 = Y_T^2 = B$, the comparison principle yields $Y_t^1 = Y_t^2$, hence $V_t^1 = V_t^2$ for every $t \in [0, T]$. This now implies

$$\begin{aligned}
\mathbb{E}\Big[\int_0^T \big|\sigma_s^{\mathrm{tr}} \varphi_s^1 - \sigma_s^{\mathrm{tr}} \varphi_s^2\big|^2 \mathrm{d}K_s\Big] &= \mathbb{E}\Big[\int_0^T |Y_s^1|^2 \, \big|\sigma_s^{\mathrm{tr}} Z_s^1 - \sigma_s^{\mathrm{tr}} Z_s^2\big|^2 \mathrm{d}K_s\Big] \\
&\le \|Y^1\|_\infty^2 \underbrace{\mathbb{E}\Big[\int_0^T \big|\sigma_s^{\mathrm{tr}} Z_s^1 - \sigma_s^{\mathrm{tr}} Z_s^2\big|^2 \mathrm{d}K_s\Big]}_{=0} = 0,
\end{aligned}$$

where uniqueness of Z^i's has been used. Due to the uniqueness of N and V, it follows that we also have $L^1 = L^2$. □

Remark 3.1.1. *In order to find solutions to* (3.5)*, it is imperative to find a good transformation reducing it to some BSDE we can handle. The key idea in the proof of Proposition 3.1.1 is the following observation: if we assume that the value process V is continuous, then for continuity to be preserved by the driver of* (3.5) *the process V must stay either in the positive or the negative half of the real line, because otherwise V in the denominator of the driver will spoil continuity. Since we deal with utility values, it is reasonable to assume V to be positive. Then it becomes reasonable to perform the logarithmic change to V, that is, we set $Y = \log(V)$ which is then well defined. An application of Itô's formula yields*

$$\begin{aligned}
\mathrm{d}Y_t &= \frac{\varphi_t^{\mathsf{tr}}}{V_t}\mathrm{d}M_t + \frac{1}{V_s}\mathrm{d}L_t - \frac{1}{2V_t^2}\mathrm{d}\langle L\rangle_t - \alpha\frac{(\varphi_t + \lambda_t V_t)^{\mathsf{tr}}}{V_t}\mathrm{d}\langle M, M\rangle_t\frac{(\varphi_t + \lambda_t V_t)}{V_t} \\
&\quad - \frac{1}{2}\frac{\varphi_t^{\mathsf{tr}}}{V_t}\mathrm{d}\langle M, M\rangle_t\frac{\varphi_t}{V_t} \\
&= \frac{\varphi_t^{\mathsf{tr}}}{V_t}\mathrm{d}M_t + \frac{1}{V_t}\mathrm{d}L_t - \frac{1}{2V_t^2}\mathrm{d}\langle L\rangle_t - \alpha\left(\frac{\varphi_t}{V_t} + \lambda_t\right)^{\mathsf{tr}}\mathrm{d}\langle M, M\rangle_t\left(\frac{\varphi_t}{V_t} + \lambda_t\right) \\
&\quad - \frac{1}{2}\frac{\varphi_t^{\mathsf{tr}}}{V_t}\mathrm{d}\langle M, M\rangle_t\frac{\varphi_t}{V_t}.
\end{aligned}$$

Setting $Z_t = \varphi_t/V_t$ and $\mathrm{d}R_t = \mathrm{d}L_t/V_t$, the above equation can be written as

$$\begin{aligned}
\mathrm{d}Y_t &= Z_t^{\mathsf{tr}}\mathrm{d}M_t + \mathrm{d}R_t - \frac{1}{2}\mathrm{d}\langle R\rangle_t \\
&\quad - \alpha\left(Z_t + \lambda_t\right)^{\mathsf{tr}}\mathrm{d}\langle M, M\rangle_t\left(Z_t + \lambda_t\right) - \frac{1}{2}Z_t^{\mathsf{tr}}\mathrm{d}\langle M, M\rangle_t Z_t,
\end{aligned}$$

which is a BSDE with a driver of quadratic growth. These BSDE have been thoroughly investigated in Lepeltier and San Martin [84], Kobylanski [77] and in Tevzadze [123] (who in contrast to the other contributors avoids transformations and tackles instead the non-linearity directly with an iteration argument on properly chosen spaces). The logarithmic transformation thus reduces the BSDE (3.5) *to a BSDE with a quadratic generator which can be handled more easily.*

3.1.3 Reduction to linear BSDEs

In the previous section, we reduced the non-standard BSDE (3.5) to a more tractable quadratic growth BSDE (3.6) using a logarithmic transformation. We can go one step further. If M exhibits the martingale representation property, we can apply an exponential transformation to the BSDE (3.6) which then reduces to a linear BSDE. The latter is known to have semi-explicit solutions, see e.g. Proposition 2.2 of El Karoui et al. [50]. The exponential change has been originally used in Kobylanski [77] and Morlais [96] to transform quadratic BSDEs into BSDEs that can be approximated by BSDEs with Lipschitz continuous drivers. In Imkeller et al. [69], this exponential coordinate change technique is used in a Brownian motion setting to transform quadratic BSDEs of the type appearing in Remark 3.1.1 into BSDEs with Lipschitz continuous drivers. In this form they become amenable to numerical approximation (more details on this can be found in Chapter 4). The composition of these two transformations then leads to the *power transformation* as detailed below. We first consider martingales M which have the predictable representation property, i.e. every one-dimensional square integrable continuous martingale N yields the representation

$$N_t = N_0 + \int_0^t Z_s^{\mathrm{tr}} \mathrm{d}M_s, \quad t \in [0,T], \tag{3.12}$$

where Z is a uniquely determined square integrable predictable process. Then we study the situation of (3.1) where the predictable representation property does not hold, and give a counterexample showing that the power transformation does not work in general.

It was pointed out to us that such a change of coordinate has already been used in Zariphopoulou [128] under the term *distortion power* in the context of reducing non-linear PDEs. Zariphopoulou [128] considers the HJB equations corresponding to an optimization problem, linearizes the dynamics via distortion and then finds a (unique) viscosity solution to the HJB equation.

Assume that condition (3.12) is in force. Then BSDE (3.5) does not contain orthogonal martingales, i.e. $L = 0$. Now this BSDE admits a unique solution which is expressed as a solution of a linear BSDE distorted by a particular exponent.

Proposition 3.1.2. *Let $\xi \in L^\infty$ be a positive $\mathcal{F}_T$-measurable random variable which is bounded away from zero. Let λ be a predictable process which satisfies $\int_0^T |\sigma_s^{\mathrm{tr}} \lambda_s|^2 \mathrm{d}K_s \le C$ almost surely for some constant $C > 0$. Let $\alpha > 0$. Then, the BSDE*

$$V_t = \xi + \alpha \int_t^T \frac{(\varphi_s + \lambda_s V_s)^{\mathrm{tr}}}{V_s} \mathrm{d}\langle M, M\rangle_s (\varphi_s + \lambda_s V_s) - \int_t^T \varphi_s^{\mathrm{tr}} \mathrm{d}M_s \tag{3.13}$$

admits a unique solution $(V, \varphi) \in \mathcal{S}^\infty \times \mathcal{H}^2$ which can be written as

$$\begin{aligned} V_t &= Y_t^{\frac{1}{2c}}, \\ \varphi_t &= \frac{Z_t}{2c} Y_t^{\frac{1}{2c}-1}, \end{aligned}$$

where $c = \alpha + \frac{1}{2}$ and $(Y, Z) \in \mathcal{S}^\infty \times \mathcal{H}^2$ is the unique solution of the linear BSDE

$$Y_t = \xi^{2c} + 2\alpha \int_t^T \left(c |\sigma_s^{\mathrm{tr}} \lambda_s|^2 Y_s + Z_s^{\mathrm{tr}} \sigma_s \sigma_s^{\mathrm{tr}} \lambda_s \right) \mathrm{d}K_s - \int_t^T Z_s^{\mathrm{tr}} \mathrm{d}M_s. \tag{3.14}$$

Proof. According to Theorem 1.1 and Proposition 2.2 from El Karoui et al. [50], the linear BSDE (3.14) admits a unique solution $(Y, Z) \in \mathcal{S}^\infty \times \mathcal{H}^2$ which is explicitly given by $Y = H^{-1}\mathbb{E}\big[\xi^{2c}H_T|\mathcal{F}_\cdot\big]$ and $Z = H^{-1}U + 2\alpha Y\lambda$ where H is given by

$$\begin{aligned} H_t &= 1 + \int_0^t 2\alpha c H_s|\sigma_s^{\mathsf{tr}}\lambda_s|^2 \mathrm{d}K_s + \int_0^t 2\alpha H_s\lambda_s^{\mathsf{tr}}\mathrm{d}M_s \\ &= 1 + \int_0^t 2\alpha c H_s|\lambda_s|^2 \mathrm{d}\langle M, M\rangle_s + \int_0^t 2\alpha H_s\lambda_s^{\mathsf{tr}}\mathrm{d}M_s \\ &= \exp\Big(\int_0^t 2\alpha|\sigma_s^{\mathsf{tr}}\lambda_s|^2\mathrm{d}K_s\Big)\mathcal{E}(2\alpha\int_0^\cdot \lambda_s^{\mathsf{tr}}\mathrm{d}M_s)_t, \quad t\in[0,T]. \end{aligned}$$

The process U is predictable and square integrable as it arises from the martingale representation

$$\mathbb{E}\big[\xi^{2c}H_T|\mathcal{F}_t\big] = \mathbb{E}[\xi^{2c}H_T] + \int_0^t U_s^{\mathsf{tr}}\mathrm{d}M_s, \quad t\in[0,T].$$

Since $\mathbb{E}\big[\exp(\int_0^t|\sigma_s^{\mathsf{tr}}\lambda_s|^2\mathrm{d}K_s)\big] \leq e^C < \infty$ for every $t\in[0,T]$, Novikov's condition is satisfied so that $\mathcal{E}(2\alpha\int_0^\cdot\lambda_s^{\mathsf{tr}}\mathrm{d}M_s)_t$ is a uniformly integrable martingale giving rise to a probability measure $\mathbb{Q} = \mathcal{E}(2\alpha\int_0^\cdot\lambda_s^{\mathsf{tr}}\mathrm{d}M_s)_T\cdot\mathbb{P}$. By $\int_0^T|\sigma_s^{\mathsf{tr}}\lambda_s|^2\mathrm{d}K_s \leq C$ it follows that $e^{2\alpha\int_t^T|\sigma_s^{\mathsf{tr}}\lambda_s|^2\mathrm{d}K_s}$ is almost surely bounded in $t\in[0,T]$. Hence there exists a constant $C>0$ such that

$$Y_t = H_t^{-1}\mathbb{E}\big[\xi^{2c}H_T|\mathcal{F}_t\big] = \mathbb{E}^{\mathbb{Q}}\big[\xi^{2c}e^{2\alpha\int_t^T|\sigma_s^{\mathsf{tr}}\lambda_s|^2\mathrm{d}K_s}|\mathcal{F}_t\big] \geq C > 0$$

holds $\mathbb{Q}$-a.s. for every $t\in[0,T]$. This means that Y is a non-negative process which is bounded away from zero $\mathbb{Q}$-a.s. But since $\mathbb{Q}\sim\mathbb{P}$, we also have that Y is $\mathbb{P}$-a.s. bounded away from zero. Note that the second equality also shows that Y is a bounded process. Therefore the processes $V = Y^{\frac{1}{2c}}$ and $\varphi = \frac{Z}{2c}Y^{\frac{1}{2c}-1}$ are well defined and by Itô's formula satisfy the BSDE

$$V_t = \xi + \alpha\int_t^T\frac{(\varphi_s+\lambda_sV_s)^{\mathsf{tr}}}{V_s}\mathrm{d}\langle M,M\rangle_s(\varphi_s+\lambda_sV_s) - \int_t^T\varphi_s^{\mathsf{tr}}\mathrm{d}M_s, \quad t\in[0,T].$$

This shows the existence of a solution. To show uniqueness, assume that (V^1,φ^1) and (V^2,φ^2) are two solutions of equation (3.13). According to Lemma 3.1.2 both value processes are bounded away from zero, hence $(Y^i,Z^i) = \big((V^i)^{2c}, 2c\frac{\varphi^i}{V^i}Y^i\big)$ for $i\in\{1,2\}$ are well defined, and by the same arguments as above, both (Y^i,Z^i) satisfy the linear BSDE (3.14). By uniqueness of the solution (Y,Z) of the linear BSDE (3.14) (see e.g. El Karoui and Huang [47]), $Y^1=Y^2$ follows which yields $V^1=V^2$. This gives rise to

$$\begin{aligned} \mathbb{E}\Big[\int_0^T|\sigma_s^{\mathsf{tr}}(\varphi_s^1-\varphi_s^2)|^2\mathrm{d}K_s\big|\mathcal{F}_t\Big] &\leq \frac{1}{4c^2}\big(\|Y\|_\infty\big)^{\frac{1}{c}-2}\,\mathbb{E}\Big[\int_0^T|\sigma_s^{\mathsf{tr}}(Z_s^1-Z_s^2)|^2\mathrm{d}K_s\big|\mathcal{F}_t\Big] \\ &= 0, \end{aligned}$$

showing that we have $\varphi^1=\varphi^2$ in $\mathcal{H}^2$. □

Now let us go back to the setting of (3.5), i.e. a scenario where the predictable representation property does not necessarily hold. If we try to extend the power transformation to (3.5) where $L \neq 0$, a formal application of Itô's formula to $Y = V^{2c}$ yields

$$Y_t = \xi^{2c} + 2\alpha \int_t^T (c\lambda_s Y_s + \lambda_s^{\mathrm{tr}} Z_s)\mathrm{d}\langle M, M\rangle_s - \int_t^T Z_s^{\mathrm{tr}}\mathrm{d}M_s - \int_t^T 2cY_s\mathrm{d}L_s + \int_t^T (c - 2c^2)Y_s\mathrm{d}\langle L, L\rangle_s. \tag{3.15}$$

This BSDE is driven by a linear generator and solved by the process triple (Y, Z, L). Now this BSDE not only requires the orthogonal martingale to be of the specific form $\int 2cY_s\mathrm{d}L_s$, but moreover contains a quadratic variation term in the orthogonal martingale which furthermore depends on the solution Y. In the following, we construct a counterexample for the case that the terminal condition ξ is unbounded.

Example 3.1.1. *Let $(\mathcal{F}_t)$ be generated by two independent one-dimensional Brownian motions W^1 and W^2. Set $M = W^1$ and $\xi = W_T^2$. Suppose that (Y, Z, L) is a solution of the zero generator BSDE*

$$Y_t = W_T^2 - \int_t^T Z_s\mathrm{d}W_s^1 - \int_t^T Y_s\mathrm{d}L_s, \quad t \in [0, T].$$

Choosing $t \in [0, T]$ and conditioning with respect to $\mathcal{F}_t$ in the last line, we get on the one hand $Y_t = W_t^2$ and on the other hand

$$Y_t = Y_0 + \int_0^t Z_s\mathrm{d}W_s^1 + \int_0^t Y_s\mathrm{d}L_s.$$

The covariation of $Y_t = W_t^2$ with $\int_0^\cdot Z_s\mathrm{d}W_s^1$ is zero, and the covariation of the right-hand side is $\int_0^\cdot Z_s^2\mathrm{d}s$, implying that $Z = 0$ almost surely. Hence,

$$Y_t = Y_0 + \int_0^t Y_s\mathrm{d}L_s = Y_0\mathcal{E}(L)_t.$$

Since $Y_0 = W_0^2 = 0$, it follows that $Y_t = 0$ which contradicts $Y_t = W_t^2$. Hence this BSDE does not have a solution.

3.1.4 Numerical tractability

The results of the previous sections essentially say that under certain conditions, the non-standard BSDE (3.5) is equivalent to the standard quadratic BSDE (3.6) or, in a setting where we have the predictable representation property, even to the linear BSDE (3.14). Now if one works in a Brownian motion setting, the task of numerically approximating (3.5) becomes much more convenient because numerical schemes for BSDEs with Lipschitz drivers have been well studied in the literature, see e.g. Bouchard and Touzi [24], Gobet et al. [54] and Bender and Denk [13]. We can use any of these schemes to solve the transformed BSDE (3.14) which by reverse transformation yields a numerical approximation of (3.13). One can even go further by approximating the quadratic BSDE (3.6) in a Brownian setting, since for this type of BSDEs, numerical approximation schemes are also available, see e.g. Imkeller and Dos Reis [66].

In view of solving (possibly high-dimensional) utility maximization problems, BSDE schemes based on Monte Carlo regression methods are a convenient and computationally efficient tool for numerical simulations. To deal with multi-dimensional problems they are particularly favorable in comparison with numerical methods for PDEs that solve corresponding HJB equations. Moreover, the description of the solution in terms of an equation is the important difference between the BSDE and the convex duality approach. Deriving solutions in the latter case that can be implemented or even deriving constructive solutions using convex duality methods remains a challenging task in general (see e.g. Kramkov and Schachermayer [80], Biagini et al. [19] and references therein). To the best of our knowledge, no computational approximations based on duality arguments exist up to date. We give several numerical examples in Section 3.3.

3.2 Applications to expected utility maximization problems

In this section, we consider several utility maximization problems all of which yield a BSDE interpretation in terms of equation (3.5). The financial market is constituted by d risky assets $S = (S^1_t, \cdots, S^d_t)^{\mathrm{tr}}_{t\in[0,T]}$ and one riskless bond which for the sake of simplicity is assumed to be of zero interest rate. Let $\lambda = (\lambda_t)_{t\in[0,T]}$ be a predictable $\mathbb{R}^{d\times 1}$-valued process which we specify at a later point. We exclude arbitrage opportunities by assuming that the set of equivalent martingale measures $\mathbb{Q} \sim \mathbb{P}$ is not empty. As before let M be an $\mathbb{R}^{d\times 1}$-valued continuous local martingale under $\mathbb{P}$ that satisfies condition (3.2). We assume that the market S evolves continuously in time, that is, S is a continuous $\mathbb{R}^{d\times 1}$-valued stochastic process governed by

$$\mathrm{d}S_t = \mathrm{d}M_t + \mathrm{d}\langle M, M\rangle_t \lambda_t, \quad t \in [0,T]. \tag{3.16}$$

On this market, private and institutional investors want to measure, control and manage risks as well as to speculate. We focus on an investor who is endowed with some initial capital $x > 0$. This investor buys and sells risky assets according to investment strategies which are $\mathbb{R}^{d\times 1}$-valued adapted stochastic processes $\pi = (\pi_t)_{t\in[0,T]}$ (π^i is the share invested in the ith stock S^i) satisfying $\mathbb{E}\left[\int_0^T |\pi_u|^2 \mathrm{d}\langle M, M\rangle_u\right] < \infty$. By $X^{x,\pi}$ we denote the wealth process of the investor associated to her initial capital and her chosen strategy (x,π),

$$X^{x,\pi} = x + \int_0^{\cdot} \pi_u^{\mathrm{tr}} \mathrm{d}S_u. \tag{3.17}$$

We say that an investment strategy π is *admissible* if in addition to square integrability, it satisfies $X^{x,\pi}_t \geq 0$ for every $t \in [0,T]$. Our aim is to study an investor whose terminal wealth is subject to a $\mathcal{F}_T$-measurable liability F and who aims at maximizing her expected utility

$$V(x) := \sup_{\pi} \mathbb{E}\left[\mathcal{U}\left(X^{x,\pi}_T, F\right)\right]. \tag{3.18}$$

Here $\mathcal{U}$ denotes some utility function modeling the preferences of the investor and which will be specified a few lines below. Due to the presence of the liability, the positivity constraint on the wealth, $X^{x,\pi} \geq 0$, needs to be modified in each of the cases that we consider in the subsequent sections. The optimization problem (3.18) admits a solution if and only if the supremum in (3.18) is attained, that is if for every $x \in \mathbb{R}_+$ there exists an admissible strategy $(\pi_t^*(x))_{0\leq t\leq T}$ such that

$$V(x) = \mathbb{E}\left[\mathcal{U}\left(X_T^{x,\pi^*(x)}, F\right)\right].$$

In the following we consider two different notions of liabilities: additive liabilities that we investigate in Section 3.2.1 and that correspond to $\mathcal{U}(x,y) \coloneqq U(x-y)$ and multiplicative liabilities that we investigate in Section 3.2.2 which correspond to $\mathcal{U}(x,y) \coloneqq U(xy)$. In both cases U denotes a deterministic utility function which we specify from case to case. More precisely, we consider utility maximization problems with respect to the power and the exponential utility and give their solutions in terms of BSDEs. We first consider optimizing a portfolio in presence of an additive liability. It turns out that for a special class of liabilities, some BSDE techniques developed for utility maximization without liability (e.g. in Hu et al. [62], Mania and Tevzadze [92]) also work in this case. In a second step, we examine a multiplicative liability in the framework of the power utility function and provide BSDE characterizations for the optimal solutions, which have also been considered in Zariphopoulou [128] and Nutz [101].

3.2.1 Additive liability

Let $U : dom(U) \to \mathbb{R}$ be a utility function that is defined on a set $dom(U) \subset \mathbb{R}$ and let $\mathcal{U}(x,y) = U(x-y)$. Then the optimization problem (3.18) can be rewritten as

$$V(0,x) \coloneqq \sup_{\pi} \mathbb{E}\left[U\left(x + \int_0^T \pi_u^{\mathsf{tr}} \mathrm{d}S_u - F\right)\right], \tag{3.19}$$

where F is a real valued and $\mathcal{F}_T$-measurable random variable which represents a liability the investor must comply with at maturity. Our approach to this type of optimization problems relies on methods developed in Hu et al. [62] and Mania and Tevzadze [92], and due to this choice, we have to restrict the class of liabilities in the following way: F is a real valued $\mathcal{F}_T$-measurable random variable which satisfies

- $\mathbb{E}[F^2] < \infty$;
- there exist a constant $c \in \mathbb{R}$ and an adapted square integrable stochastic process $(\eta_t)_{t\in[0,T]}$ in $\mathbb{R}^{d\times 1}$ such that

$$F = c + \int_0^T \eta_u^{\mathsf{tr}} \mathrm{d}S_u. \tag{3.20}$$

If this represents the class of all possible liabilities, then condition (3.20) means that every claim F is replicated by the process η, implying that we are in the setting of a complete market. Since U is *a priori* defined on $dom(U)$, we define the set of optimal strategies Π_x^η by

$$\Pi_x^\eta := \Big\{\pi : \Omega \times [0,T] \to \mathbb{R}^d : \ \pi \text{ adapted}, \ \mathbb{E}\Big[\int_0^T |\pi_u|^2 \mathrm{d}\langle M, M\rangle_u\Big] < \infty,$$
$$x + \int_0^t (\pi_u - \eta_u)^{\mathsf{tr}} \mathrm{d}S_u \in dom(U) \ \forall t \in [0,T], \mathbb{P} - a.s.\Big\}.$$

Denote for convenience $x^{(F)} := x - c$ and $\Pi_x := \Pi_x^0$ for $x > 0$. Liabilities satisfying (3.20) allow the reduction of problem (3.19) to a portfolio optimization problem which does not involve liabilities. To this end, we consider the dynamical version of (3.19)

$$V(t,x) := \operatorname{esssup}_{\pi \in \Pi^\eta_{x^{(F)}}} \mathbb{E}\left[U\left(x^{(F)} + \int_t^T (\pi_u - \eta_u)^{\mathsf{tr}} \mathrm{d}S_u\right) \Big| \mathcal{F}_t\right].$$

Then it follows that

$$\begin{aligned} V(t,x) &= \operatorname{esssup}_{\pi \in \Pi^\eta_{x^{(F)}}} \mathbb{E}\left[U\left(x^{(F)} + \int_t^T (\pi_u - \eta_u)^{\mathsf{tr}} \mathrm{d}S_u\right) \Big| \mathcal{F}_t\right] \\ &= \operatorname{esssup}_{\tilde{\pi} \in \Pi_{x^{(F)}}} \mathbb{E}\left[U\left(x^{(F)} + \int_t^T \tilde{\pi}_u^{\mathsf{tr}} \mathrm{d}S_u\right) \Big| \mathcal{F}_t\right], \end{aligned} \tag{3.21}$$

where the second equality results from the identity

$$\Pi^\eta_{x^{(F)}} = \Pi_{x^{(F)}} + \eta = \left\{\rho : \Omega \times [0,T] \to \mathbb{R}^d, \ \rho = \pi + \eta, \ \pi \in \Pi_{x^{(F)}}\right\}.$$

Equation (3.21) exemplifies the reduction of (3.19) to an easier problem and underlines the imperative character of condition (3.20): it allows to merge the liability into the set of admissible strategies by an affine shift. Let us summarize this relationship between optimizing with and without liability.

Lemma 3.2.1. *Assume that the problem*

$$esssup_{\tilde{\pi} \in \Pi_{x^{(F)}}} \mathbb{E}\left[U\left(x^{(F)} + \int_t^T \tilde{\pi}_u^{\mathsf{tr}} \mathrm{d}S_u\right) \Big| \mathcal{F}_t\right]$$

admits an optimal strategy $\tilde{\pi}^* \in \Pi_{x^{(F)}}$. *Then* $\pi^* := \tilde{\pi}^* + \eta$ *is an optimal strategy for* (3.21).

Let us now study the reformulation of (3.19) for the power and exponential case in more details. We will see that the optimal solutions yield an explicit representation in terms of the BSDE (3.5).

Power utility

We derive the solution of the optimization problem (3.19) under the power utility function $U(x) = x^\gamma$, $\gamma \in (0,1)$. This solution is characterized by the BSDE (3.5). Using power utility, equation (3.19) becomes

$$V(x) := \sup_{\pi\in\Pi_x^\eta} \mathbb{E}\left[\left(x + \int_0^T \pi_u^{\mathrm{tr}} \mathrm{d}S_u - F\right)^\gamma\right],$$

where F satisfies (3.20) and the set of admissible strategies is given by

$$\Pi_x^\eta := \Big\{\pi : \Omega \times [0,T] \to \mathbb{R}^d : \ \pi \text{ is } (\mathcal{F}_t)\text{-adapted}, \\ x + \int_0^t (\pi_u - \eta_u)^{\mathrm{tr}} \mathrm{d}S_s \geq 0 \ \forall t \in [0,T], \mathbb{P} - a.s.\Big\}.$$

We assume that x and c are such that $x^{(F)} = x - c \geq 0$. Introducing $\tilde{\pi} := \pi - \eta$ and $\hat{\pi} := \frac{\tilde{\pi}}{x}$, it is obvious that they are mutually related by

$$\pi \in \Pi_{x^{(F)}}^\eta \Leftrightarrow \tilde{\pi} := \pi - \eta \in \Pi_{x^{(F)}} \Leftrightarrow \hat{\pi} := \frac{\tilde{\pi}}{x^{(F)}} \in \Pi_1.$$

Hence Lemma 3.2.1 yields

$$\begin{aligned} V(t,x) &:= \mathrm{esssup}_{\pi\in\Pi_{x^{(F)}}^\eta} \mathbb{E}\left[\left(x^{(F)} + \int_t^T (\pi_u - \eta_u)^{\mathrm{tr}} \mathrm{d}S_u\right)^\gamma \middle| \mathcal{F}_t\right] \\ &= \mathrm{esssup}_{\tilde{\pi}\in\Pi_{x^{(F)}}} \mathbb{E}\left[\left(x^{(F)} + \int_t^T \tilde{\pi}_u^{\mathrm{tr}} \mathrm{d}S_u\right)^\gamma \middle| \mathcal{F}_t\right] \\ &= (x^{(F)})^\gamma \mathrm{esssup}_{\hat{\pi}\in\Pi_1} \mathbb{E}\left[\left(1 + \int_t^T \hat{\pi}_u^{\mathrm{tr}} \mathrm{d}S_u\right)^\gamma \middle| \mathcal{F}_t\right] \\ &= (x^{(F)})^\gamma V_t, \end{aligned}$$

where

$$V_t := \mathrm{esssup}_{\hat{\pi}\in\Pi_1} \mathbb{E}\left[\left(1 + \int_t^T \hat{\pi}_u^{\mathrm{tr}} \mathrm{d}S_u\right)^\gamma \middle| \mathcal{F}_t\right] \tag{3.22}$$

has the terminal condition $V_T = 1$. It is straightforward to see that $(V_t)_{t\in[0,T]}$ is a supermartingale (see e.g. Section 4 of Mania and Tevzadze [92]). By the Galtchouk-Kunita-Watanabe (GKW) decomposition for V, there exists a predictable one-dimensional finite variation process A, an adapted stochastic process φ in $\mathbb{R}^{d\times 1}$ and a one-dimensional square integrable martingale L strongly orthogonal to M such that

$$\begin{aligned} V_t &= V_0 + A_t + \int_0^t \varphi_s^{\mathrm{tr}} \mathrm{d}M_s + L_t \\ &= 1 - \int_t^T \mathrm{d}A_s - \int_t^T \varphi_s^{\mathrm{tr}} \mathrm{d}M_s - \int_t^T \mathrm{d}L_s, \quad t \in [0,T], \end{aligned}$$

i.e. V is a BSDE. The following result gives an explicit representation of the finite variation process A in the GKW representation. Once we get the BSDE for V it becomes straightforward to give a closed form expression for the optimal strategy in terms of this BSDE. The proof essentially makes use of Lemma 3.2.1 to transform the optimization problem with liability into one without. The latter can then be treated by Theorems 3.1 and 4.1 from Mania and Tevzadze [92].

Lemma 3.2.2. *The process V from equation* (3.22) *satisfies the BSDE*

$$\begin{aligned} V_t = 1 - \frac{q}{2}\int_t^T \frac{(\varphi_s + \lambda_s V_s)^{\mathsf{tr}}}{V_s} \mathrm{d}\langle M, M\rangle_s (\varphi_s + \lambda_s V_s) \\ - \int_t^T \varphi_s^{\mathsf{tr}} \mathrm{d}M_s - (L_T - L_t), \end{aligned} \tag{3.23}$$

where $q := \frac{\gamma}{\gamma - 1}$. Moreover, the optimal strategy is given by

$$\pi_t^* = -x^{(F)}(q-1)\left(\frac{\varphi_t}{V_t} + \lambda_t\right)\mathcal{E}\left(-(q-1)\int_0^\cdot \left(\frac{\varphi_u}{V_u} + \lambda_u\right)^{\mathsf{tr}} \mathrm{d}S_u\right)_t + \eta_t,$$

and the associated optimal wealth process $X^{x^{(F)},\pi^*}$ by

$$X_t^{x^{(F)},\pi^*} = x^{(F)}\mathcal{E}\left(-(q-1)\int_0^\cdot \left(\frac{\varphi_u}{V_u} + \lambda_u\right)^{\mathsf{tr}} \mathrm{d}S_u\right)_t + \int_0^t \eta_u^{\mathsf{tr}} \mathrm{d}S_u.$$

Proof. By Lemma 3.2.1 we have to show that the optimization problem $V(t,x) = (x^{(F)})^\gamma V_t$ admits an optimal strategy $\tilde{\pi}^*$ and that this optimal strategy can be characterized in terms of a BSDE. Since the power utility function satisfies the asymptotic elasticity condition, *i.e.*

$$\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1,$$

the existence of an optimal strategy $\tilde{\pi}^*$ is guaranteed (see e.g. Kramkov and Schachermayer [80]). Thus all the hypotheses from Mania and Tevzadze [92, Theorem 3.1, Theorem 4.1] are satisfied, entailing as a consequence that V is a solution of equation (3.23). Now we denote by $X^{x^{(F)},\tilde{\pi}^*}$ the wealth process associated to the optimal strategy

$$\tilde{\pi}_t^* = \mathrm{argsup}_{\tilde{\pi}\in\Pi_{x^{(F)}}} \mathbb{E}\left[\left(x^{(F)} + \int_t^T \tilde{\pi}_u^{\mathsf{tr}} \mathrm{d}S_u\right)^\gamma \middle| \mathcal{F}_t\right]$$

which is given by

$$X_t^{x^{(F)},\tilde{\pi}^*} = x^{(F)} + \int_0^t (\tilde{\pi}_u^*)^{\mathsf{tr}} \mathrm{d}S_u, \quad t \in [0,T].$$

By Mania and Tevzadze [92, Theorem 4.1], we have for $t \in [0,T]$

$$X_t^{x^{(F)},\tilde{\pi}^*} = x^{(F)}\mathcal{E}\left(-(q-1)\int_0^\cdot \left(\frac{\varphi_u}{V_u} + \lambda_u\right)^{\mathsf{tr}} \mathrm{d}S_u\right)_t,$$

which implies

$$\tilde{\pi}_t^* = -x^{(F)}(q-1)\left(\frac{\varphi_t}{V_t} + \lambda_t\right)\mathcal{E}\left(-(q-1)\int_0^\cdot \left(\frac{\varphi_u}{V_u} + \lambda_u\right)^{\mathrm{tr}} \mathrm{d}S_u\right)_t .$$

Then the identity

$$\begin{aligned}\pi_t^* &= \operatorname{argsup}_{\pi \in \Pi^\eta_{x^{(F)}}} \mathbb{E}\left[\left(x^{(F)} + \int_t^T (\pi_u - \eta_u)\mathrm{d}S_u\right)^\gamma \middle| \mathcal{F}_t\right] \\ &= \tilde{\pi}_t^* + \eta_t\end{aligned}$$

yields the claim. □

Obviously equation (3.23) has a unique solution by Proposition 3.1.1 because it belongs to the class of BSDEs of type (3.5).

Remark 3.2.1 (Mean variance hedging)**.** *If we consider other values for γ, for instance $\gamma > 1$, Lemma 3.2.2 (applied to $-U$) remains true. Observe that if $\gamma = 2$, there is a relationship between the mean variance hedging problem with a constant liability $b > 0$ and utility maximization with respect to the utility function $U(x) = 2bx - x^2 = b^2 - (x-b)^2$ (see also Section 4 in Mania and Tevzadze [92]). More precisely, the problem of minimizing the hedging error via*

$$\mathit{essinf}_\pi \mathbb{E}\Big[(x + \int_t^T \pi_u^{\mathrm{tr}} \mathrm{d}S_u - b)^2 \Big| \mathcal{F}_t\Big]$$

is equivalent to maximizing

$$\begin{aligned}\mathit{esssup}_\pi \mathbb{E}\Big[U(x + \int_0^T \pi^{\mathrm{tr}}\mathrm{d}S_u)\Big|\mathcal{F}_t\Big] &= \mathit{esssup}_\pi \mathbb{E}\Big[2b(x + \int_0^T \pi^{\mathrm{tr}}\mathrm{d}S_u) - (x + \int_0^T \pi^{\mathrm{tr}}\mathrm{d}S_u)^2\Big|\mathcal{F}_t\Big] \\ &= b^2 - \mathit{essinf}_\pi \mathbb{E}\Big[(x + \int_0^T \pi^{\mathrm{tr}}\mathrm{d}S_u - b)^2\Big|\mathcal{F}_t\Big] \\ &= b^2 - \mathit{essinf}_\pi \mathbb{E}\Big[(x-b)^2\Big(1 + \int_0^T (\frac{\pi}{x-b})^{\mathrm{tr}}\mathrm{d}S_u\Big)^2\Big|\mathcal{F}_t\Big] \\ &= b^2 + (x-b)^2 \mathit{esssup}_\pi \mathbb{E}\Big[\Big(1 + \int_0^T \tilde{\pi}^{\mathrm{tr}}\mathrm{d}S_u\Big)^2\Big|\mathcal{F}_t\Big] \\ &= b^2 + (x-b)^2 V_t,\end{aligned}$$

where V_t satisfies the BSDE

$$\begin{aligned}V_t = 1 - \int_t^T &\frac{(\varphi_s + \lambda_s V_s)^{\mathrm{tr}}}{V_s}\mathrm{d}\langle M, M\rangle_s\,(\varphi_s + \lambda_s V_s) \\ &- \int_t^T \varphi_s^{\mathrm{tr}}\mathrm{d}M_s - (L_T - L_t),\end{aligned}$$

which is identical to equation (3.23) *for $q = 2$.*

Exponential utility

In this section we discuss the case $U(x) = -e^{-\alpha x}$ for $x \in \mathbb{R}$ with some fixed risk aversion parameter $\alpha > 0$. In Hu et al. [62] the optimization problem (3.19) has already been solved for general bounded liabilities. We extend some results from Hu et al. [62] by providing a solution for liabilities which are not necessarily bounded. Since this section has only a marginal connection to the BSDE (3.5), our approach is of rather illustrative character which is the reason to make a few simplifications: we assume throughout this section that we have $\mathrm{d}S_t = \sigma_t \mathrm{d}W_t + b_t \mathrm{d}t$ where σ is a $\mathbb{R}^{d\times d}$-valued adapted process, b is a $\mathbb{R}^{d\times 1}$-valued adapted process and W denotes a d-dimensional Brownian motion. Assuming that $\sigma\sigma^{\mathsf{tr}}$ is invertible, we consider $\mathrm{d}M_t = \sigma_t \mathrm{d}W_t$ and $\lambda_t := (\sigma_t\sigma_t^{\mathsf{tr}})^{-1} b_t$ in (3.16). These dynamics of the price process reproduce the setup from Hu et al. [62]. Note that the results presented here can be easily extended to the general continuous semimartingale setting. We refer to Section 3.2.2 where the approaches by Hu et al. [62] and Morlais [96] are described in a more general framework. Let us first recall the main result of Hu et al. [62] which considers bounded liabilities.

Bounded liability case: Let the liability F be a bounded $\mathcal{F}_T$-measurable random variable satisfying (3.20). Furthermore, assume the investor can only employ strategies π which belong to a closed set $\tilde{C}$ of $\mathbb{R}^d$. Constraints on strategies appear often in reality and reflect e.g. government regulations or companies' internal risk management policies. Note that this setting is a particular case of Hu et al. [62], and we refer the reader to it for considering stocks of lognormal type. For convenience we let $p_t := \pi_t^{\mathsf{tr}}\sigma_t$ (so that the constraint $\pi_t \in \tilde{C}$ becomes $p_t \in C_t$ with $C_t := \tilde{C}\sigma_t$ for every $t \in [0,T]$). With this notation, we let $X_t^{x,p} := x + \int_0^t p_s \mathrm{d}W_s + \int_0^t p_s\theta_s \mathrm{d}s$ with $\theta_s := \sigma_s^{\mathsf{tr}}\lambda_s$. The set of admissible strategies for the investor is then given by

$$\Pi^b := \Big\{ p : \Omega \times [0,T] \to \mathbb{R}^d : \ p \text{ is adapted and } \\ \mathbb{E}\Big[\int_0^T |p_u|^2 \mathrm{d}u\Big] < \infty, \ p_t \in C_t \ \forall t \in [0,T] \Big\}.$$

Using this setup, it has been shown in Hu et al. [62] that the optimization problem $V^b(x)$, $x > 0$, defined by

$$V^b(x) := -\sup_{p\in\Pi^b} \mathbb{E}\left[\exp\left(-\alpha(X_T^{x,p} - F)\right)\right],$$

admits at least one optimal p^* such that every time t, p_t^* is given as the projection of a process Z_t^b onto the set C_t, i.e. $p_t^* = \mathrm{proj}(Z_t^b + \frac{\theta_t}{\alpha}, C_t)$ where (Y^b, Z^b) denotes the unique solution of the BSDE

$$Y_t^b = F - \int_t^T (Z_s^b)^{\mathsf{tr}} \mathrm{d}W_s - \int_t^T f^b(s, Z_s^b)\mathrm{d}s, \quad t \in [0,T], \tag{3.24}$$

with $f^b(s,z) := -\frac{\alpha}{2} dist^2\left(z + \frac{\theta_s}{\alpha}, C_s\right) + z^{\mathsf{tr}}\theta_s + \frac{|\theta_s|^2}{2\alpha}$, $z \in \mathbb{R}^d$. In addition, the value function is given by $V^b(x) = -\exp(-\alpha(x - Y_0^b))$, $x > 0$. Before turning to the unbounded case we state two remarks which will be of importance in the following.

Remark 3.2.2. *The existence and uniqueness of the BSDE* (3.24) *is due to the boundedness assumption on F, since the driver f^b has quadratic growth in the z-variable. Indeed, a classical result from Kobylanski [77, Theorem 2.3] provides existence, while uniqueness has been proved in Hu et al. [62, Proof of Theorem 7].*

Remark 3.2.3. *Notably in the works Briand and Hu [26, 27] and Briand et al. [29], the boundedness condition on F has been relaxed to some exponential moment conditions which are essential to prove uniqueness of a solution. We refer to reader to Briand and Hu [27] where uniqueness is proved for convex drivers (like the one we are considering) and to Ankirchner et al. [5] where a number of counterexamples to uniqueness are constructed in the case $f(z) = z^2$.*

Unbounded liability case: Assume that F is a square integrable $\mathcal{F}_T$-measurable random variable satisfying (3.20). Admissible strategies p_t will be understood as processes taking their values in a closed set C_t (again of the form $\tilde{C}\sigma_t$) in $\mathbb{R}^d$ for $t \in [0,T]$. Since the liability now is unbounded (and *a priori* can have infinite exponential moments), the investor's strategies p_t are constrained to the set $C_t + \eta_t$ for every $t \in [0,T]$. In other words, due to F being unbounded, the investor is allowed to escape the formal constraint sets C_t, yet this escape is subject to an amount determined by η_t. If the investor is a trader in a company, then one could see our setting as an extension of the usual internal regulations modeled by the sets C_t in order to hedge this unbounded liability. Taking these remarks into account, we allow the set of admissible strategies to be given by

$$\begin{aligned}\Pi := \Big\{ p : \Omega \times [0,T] \to \mathbb{R}^d :\ & p \text{ is adapted and} \\ & \mathbb{E}\Big[\int_0^T |p_u|^2 \mathrm{d}u\Big] < \infty,\ p_t \in C_t + \eta_t \ a.s.\ \forall t \in [0,T]\Big\}.\end{aligned}$$

We also introduce another set of strategies

$$\begin{aligned}\tilde{\Pi} = \Pi - \eta := \Big\{ \tilde{p} : \Omega \times [0,T] \to \mathbb{R}^d :\ & \tilde{p} \text{ adapted and} \\ & \mathbb{E}\Big[\int_0^T |\tilde{p}_u|^2 \mathrm{d}u\Big] < \infty,\ \tilde{p}_t \in C_t \ a.s.\ \forall t \in [0,T]\Big\}.\end{aligned}$$

In this notation, the investor's optimization problem is

$$\begin{aligned} V(x) &:= \sup_{p \in \Pi} \mathbb{E}\big[-\exp(-\alpha(X_T^{x,p} - F))\big] \qquad (3.25) \\ &= \sup_{p \in \Pi} \mathbb{E}\Big[-\exp\Big(-\alpha\Big(x^{(F)} + \int_0^T (p_u - \eta_u)^{\mathrm{tr}} \mathrm{d}S_u\Big)\Big)\Big] \\ &= \sup_{\tilde{p} \in \tilde{\Pi}} \mathbb{E}\Big[-\exp\Big(-\alpha\Big(x^{(F)} + \int_0^T \tilde{p}_u^{\mathrm{tr}} \mathrm{d}S_u\Big)\Big)\Big], \qquad (3.26) \end{aligned}$$

where $x^{(F)} = x - c$. In other words, one can replace an optimization problem of type (3.25) that has a liability F and trading restrictions given by Π by an optimization problem of type (3.26) that has no liability but trading restrictions translated by η, the replication process of F.

Lemma 3.2.3. *Let F be square integrable liability as above. Then, an optimal strategy p^* of (3.25) is given by $p^*_t = \mathrm{proj}(Z^F_t - \eta_t + \frac{\theta_t}{\alpha}, C_t) + \eta_t, t \in [0,T]$, where (Y^F, Z^F) is the unique solution of the BSDE*

$$Y^F_t = F - \int_t^T (Z^F_s)^{\mathrm{tr}} \mathrm{d}W_s - \int_t^T f^F(s, Z^F_s)\mathrm{d}s, \tag{3.27}$$

with $f^F(s,z) := -\frac{\alpha}{2} dist^2\left(z - \eta_s + \frac{\theta_s}{\alpha}, C_s\right) + \frac{|\theta_s|^2}{2\alpha} + (z^{\mathrm{tr}} - \eta_s)\theta_s$ for $s \in [0,T]$ and $z \in \mathbb{R}^d$.

Before entering into the details note that the driver of BSDE (3.27) is quadratic in z. From the literature we know that existence and uniqueness of solutions for such a BSDE are ensured if the terminal condition F is bounded or has at least finite exponential moments. Here, we are able to show existence and uniqueness without this assumption. This is due to the particular form of the driver which in a sense takes into account the terminal condition F. We refer the interested reader to Remark 3.2.4.

Proof of Lemma 3.2.3. It is shown in Hu et al. [62, Theorem 7] that a solution $\tilde{p}^*$ of (3.26) exists and is given as the projection of Z^0 on the set C, i.e. $\tilde{p}^*_t := \mathrm{proj}(Z^0_t + \frac{\theta_t}{\alpha}, C)$ where (Y^0, Z^0) is solution of the BSDE

$$Y^0_t = 0 - \int_t^T (Z^0_s)^{\mathrm{tr}} \mathrm{d}W_s - \int_t^T f^0(Z^0_s)\mathrm{d}s, \quad 0 \le t \le T, \tag{3.28}$$

with $f^0(s,z) := -\frac{\alpha}{2} dist^2\left(z + \frac{\theta_s}{\alpha}, C_s\right) + z^{\mathrm{tr}}\theta_s + \frac{|\theta_s|^2}{2\alpha}$. From the classical result of Kobylanski [77, Theorem 2.3] the BSDE (3.28) admits at least a solution, and by Hu et al. [62, Theorem 7] uniqueness is guaranteed. Since $\tilde{\Pi} = \Pi - \eta$ we get from Theorem 3.2.1 that $p^* := \tilde{p}^* + \eta = \mathrm{proj}(Z^0 + \frac{\theta}{\alpha}, C) + \eta$ is an optimal strategy for (3.26). Existence and uniqueness of the solution of BSDE (3.28) imply that a unique solution of (3.27) exists and is given by $Y^F = Y^0 + \int_0^\cdot \eta_u \mathrm{d}W_u$ and $Z^F = Z^0 + \eta$. Indeed let $U := Y^0 + \int_0^\cdot \eta_u \mathrm{d}W_u$ and $V := Z^0 + \eta$. Then equation (3.28) implies

$$\begin{aligned} U_t &= F - \int_t^T V_s^{\mathrm{tr}} \mathrm{d}W_s - \int_t^T f^0(Z^0_s)\mathrm{d}s \\ &= F - \int_t^T V_s^{\mathrm{tr}} \mathrm{d}W_s - \int_t^T f^F(s, V_s)\mathrm{d}s, \quad t \in [0,T], \end{aligned}$$

where the last equality comes from the fact that

$$f^F(s, V_s) = -\frac{\alpha}{2} dist^2\left(Z^0_s + \frac{\theta_s}{\alpha}, C_s\right) + \frac{|\theta_s|^2}{2\alpha} + (Z^0_s)^{\mathrm{tr}}\theta_s = f^0(s, Z^0_s), \quad s \in [0,T].$$

This proves that a solution of (3.27) exists. Its uniqueness is a direct consequence of the previous computation and the uniqueness of the solution of BSDE (3.28). □

Remark 3.2.4. *As a by-product, we get that the quadratic BSDE* (3.27) *admits a unique solution with terminal condition F which is neither assumed to be bounded nor to have finite exponential moments. Obviously the quadratic driver f^F has a special form, since it contains the terminal condition F via the predictable process η. This type of driver escapes and complements the analysis of Briand and Hu [26, 27], Briand et al. [29].*

3.2.2 Multiplicative liability for power utility

In this section we derive a BSDE for solving (3.18) for the case that U is the power utility function $U(x) = x^\gamma$ with $x > 0$ for some fixed $\gamma \in (0,1)$. Our objective now is to solve the optimization problem

$$V(x) = \sup_{\pi\in\Pi^x} \mathbb{E}\left[(X_T^{x,\pi})^\gamma F^\gamma\right] \tag{3.29}$$

with $X^{x,\pi}$ given by (3.17) and F being an $\mathcal{F}_T$-measurable random variable satisfying $0 < F < 1$. Note that such liabilities assess the wealth of the investor by a random proportion at maturity $T > 0$. One can think of F as some proportion of charges or taxes which are subject to external fluctuations. In order to solve (3.29), let $\bar{\rho}_t^i$ be the part of the wealth invested in the i-th stock at time t. We denote $\rho_t^i = \frac{\bar{\rho}_t^i}{S_t^i}$ and by ρ we denote the vector in $\mathbb{R}^{d\times 1}$ with i-th component ρ^i for $i = 1,\dots,d$. With this parametrization of the strategies, the wealth process satisfies

$$X_t^{x,\rho} = x + \int_0^t X_u^{x,\rho}\rho_u \mathrm{d}S_u = x\exp\left(\int_0^t \rho_u^{\mathsf{tr}}\mathrm{d}S_u - \frac{1}{2}\int_0^t \rho_u^{\mathsf{tr}}\mathrm{d}\langle M,M\rangle_u\rho_u\right).$$

We assume that our investor has to face trading constraints (coming for example from a regulator) modeled by a closed, not necessarily convex set C in $\mathbb{R}^d$. The set of admissible strategies is then given by the square integrable stochastic processes $(\rho_t)_{t\in[0,T]}$ such that ρ_t belongs $\mathbb{P}$-almost surely to C for every t. As a consequence, (3.29) becomes

$$V(x) = \sup_{\rho\in\Pi^C} \mathbb{E}\left[x^\gamma \exp\left(\gamma\int_0^t \rho_u^{\mathsf{tr}}\mathrm{d}S_u - \frac{\gamma}{2}\int_0^t \rho_u^{\mathsf{tr}}\mathrm{d}\langle M,M\rangle_u\rho_u\right)F^\gamma\right] \tag{3.30}$$

with

$$\Pi^C := \Big\{\rho : \Omega\times[0,T] \to \mathbb{R}^d : \rho \text{ is adapted and } \mathbb{E}\Big[\int_0^T \rho_u^{\mathsf{tr}}\mathrm{d}\langle M,M\rangle_u\rho_u\Big] < \infty \text{ and } \rho_t \in C \ a.s.\ \forall t\in[0,T]\Big\}.$$

Our approach to solve (3.29) is a straightforward modification of the computations from Hu et al. [62, Section 3] (see also Morlais [96] for the continuous martingale setting). We nevertheless repeat here the essential ingredients. The key ingredient of martingale optimality comes down to the task of finding a family of stochastic processes $R^{x,\rho}$ such that

1. $R_T^{x,\rho} = U(X_T^{x,\rho}F)$ for all ρ in Π_x;

2. $R_0^{x,\rho} = R_0$ is constant for all ρ in Π_x;
3. $R^{x,\rho}$ is a supermartingale for all ρ in Π_x and there exists an element ρ^* in Π_x such that R^{x,ρ^*} is a martingale.

The formulation of the problem in (3.30) suggests that the process R is of the form

$$R_t^{x,\rho} := x^\gamma \exp\left(\gamma \int_0^t \rho_u^{\mathsf{tr}} \mathrm{d}S_u - \frac{\gamma}{2}\int_0^t \rho_u^{\mathsf{tr}} \mathrm{d}\langle M, M\rangle_u \rho_u\right) \exp(Y_t),$$

where Y solves a BSDE

$$Y_t = \gamma \log(F) - \int_t^T Z_s^{\mathsf{tr}} \mathrm{d}M_s - \int_t^T f(s, Z_s)\mathrm{d}K_s - \int_t^T \mathrm{d}L_s + \frac{1}{2}\int_t^T \mathrm{d}\langle L, L\rangle_s, \tag{3.31}$$

with a driver f yet to be determined. This way, $R^{x,\rho}$ can be rewritten as

$$R_t^{x,\rho} = x^\gamma \exp(Y_0)\mathcal{E}\left(\int_0^\cdot (\gamma\rho_u + Z_u)^{\mathsf{tr}} \mathrm{d}M_u + L_t\right)_t \exp\left(I_t^{x\rho}\right).$$

Recalling $\mathrm{d}\langle M, M\rangle_t = \sigma_t \sigma_t^{\mathsf{tr}} \mathrm{d}K_t$, we have

$$I_t^{x\rho} := \int_0^t \left[\frac{1}{2}|\sigma_u^{\mathsf{tr}}\left(\gamma\rho_u + Z_u\right)|^2 - \frac{\gamma}{2}|\sigma_u^{\mathsf{tr}}\rho_u|^2 + \gamma(\rho_u^{\mathsf{tr}}\sigma_u\sigma_u^{\mathsf{tr}}\lambda_u) + f(u, Z_u)\right]\mathrm{d}K_u.$$

Now martingale optimality requires to look for drivers f such that for every ρ we have

$$\frac{1}{2}|\sigma_u^{\mathsf{tr}}\left(\gamma\rho_u + Z_u\right)|^2 - \frac{\gamma}{2}|\sigma_u^{\mathsf{tr}}\rho_u|^2 + \gamma(\rho_u^{\mathsf{tr}}\sigma_u\sigma_u^{\mathsf{tr}}\lambda_u) + f(u, Z_u) \le 0,\ u \in [0, T], \tag{3.32}$$

and such that there exists a ρ^* for which the inequality above becomes an equality. According to (3.32) we need

$$f(u, Z_u) \le \frac{\gamma(1-\gamma)}{2}\left|\sigma_u^{\mathsf{tr}}\rho_u - \frac{\sigma_u^{\mathsf{tr}}(Z_u + \lambda_u)}{1-\gamma}\right|^2 - \frac{\gamma}{2(1-\gamma)}|(Z_u + \lambda_u)^{\mathsf{tr}}\sigma_u|^2 - \frac{1}{2}|\sigma_u^{\mathsf{tr}} Z_u|^2,$$

which leads to the candidate

$$f(u, z) = \frac{\gamma(1-\gamma)}{2}\ dist^2\left(\frac{\sigma_u^{\mathsf{tr}}(z + \lambda_u)}{1-\gamma}, \tilde{C}_u\right) - \frac{\gamma}{2(1-\gamma)}|(z + \lambda_u)^{\mathsf{tr}}\sigma_u|^2 - \frac{1}{2}|\sigma_u^{\mathsf{tr}} z|^2, \tag{3.33}$$

with $\tilde{C}_u := \sigma_u^{\mathsf{tr}} C$ and

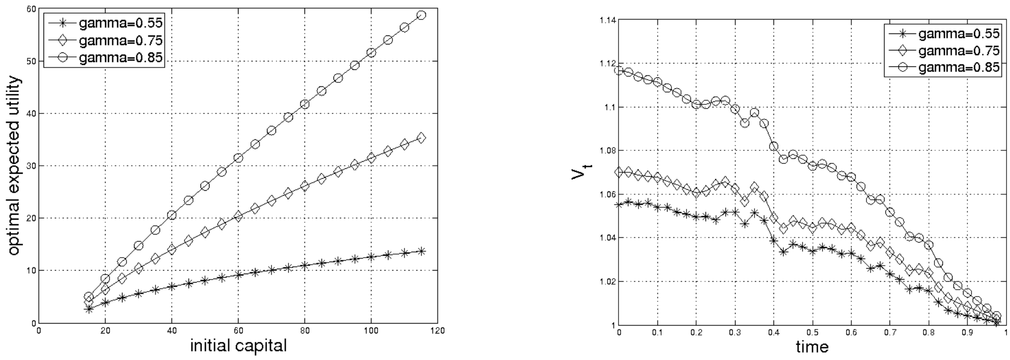
$$dist^2\left(\frac{(z + \lambda_u)}{1-\gamma}, \tilde{C}_u\right) := \min_{\tilde{\rho}_u \in \tilde{C}_u}\left|\tilde{\rho}_u - \frac{\sigma_u^{\mathsf{tr}}(Z_u + \lambda_u)}{1-\gamma}\right|^2, \quad (u, z) \in [0, T] \times \mathbb{R}^d.$$

Then, the optimal strategy ρ^* is given by $\tilde{\rho}^* = \sigma^{\mathsf{tr}}\rho^*$ with $\tilde{\rho}^*$ an element realizing the distance above.

Remark 3.2.5. *Note that if no constraints are imposed on the strategies (i.e. $C = \mathbb{R}^d$)*

the driver of equation (3.33) *becomes*

$$f(u,z) = -\frac{\gamma}{2(1-\gamma)}|(z+\lambda_u)\sigma_u|^2 - \frac{1}{2}|\sigma_u^{\mathrm{tr}} z|^2, \quad (u,z) \in [0,T] \times \mathbb{R}^d.$$



(a) Optimal expected utility $V(0,x)$ in dependence of the initial capital x.

(b) Pathwise supermartingale property of the BSDE value process V.

Figure 3.1: Optimal expected utility and pathwise supermartingale plot.

3.3 Numerical simulation of utility maximization problems

To illustrate the algorithm presented in the previous sections, we provide two numerical examples on solving the additive and multiplicative maximization problem with respect to the power utility function. Moreover, we provide a third example dealing with optimization with respect to the exponential utility function under integer constraints.

3.3.1 Example for the additive power utility case

Recall the additive optimization problem from Section 3.2.1,

$$V(x) = \sup_{\pi \in \Pi_x^\eta} \mathbb{E}\left[\left(x + \int_0^T \pi_u^{\mathrm{tr}} \mathrm{d}S_u - F\right)^\gamma\right],$$

where $x > 0$ denotes the initial capital, $\gamma \in (0,1)$ denotes the risk aversion parameter and the $\mathcal{F}_T$-measurable bounded liability F satisfies $F = c + \int_0^T \eta_u \mathrm{d}S_u$ for a constant $c > 0$ and a predictable process η. We consider a one-dimensional Black-Scholes market composed of a stock and a zero interest rate bank account. The stock evolves according to

$$\mathrm{d}S_t = \sigma S_t \mathrm{d}W_t + \mu S_t \mathrm{d}t,$$

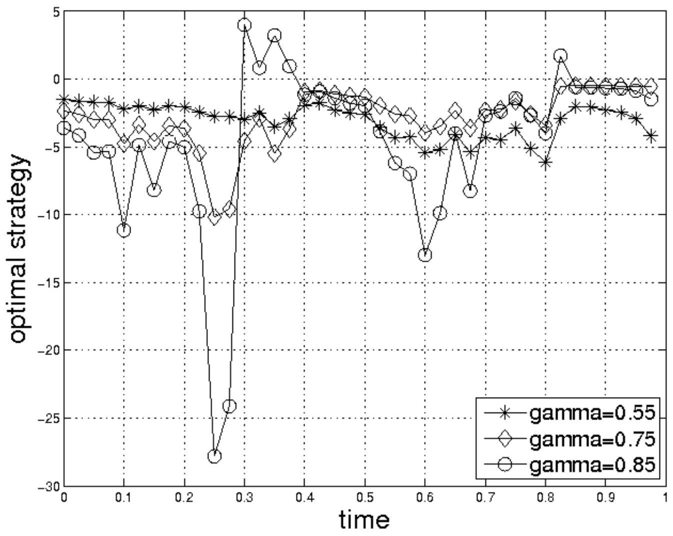


Figure 3.2: Sample paths of the wealth and the optimal investment strategy at different risk aversion levels of γ.

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are the constant drift and volatility coefficients. Putting $\mathrm{d}M_t := \sigma S_t \mathrm{d}W_t$ and

$$\lambda_t := \frac{\mu S_t}{\sigma^2 S_t^2} = \frac{\mu}{\sigma^2} S_t^{-1}, \tag{3.34}$$

we recover the martingale M and the market price of risk process λ. In this setting, the BSDE (3.23) reduces to

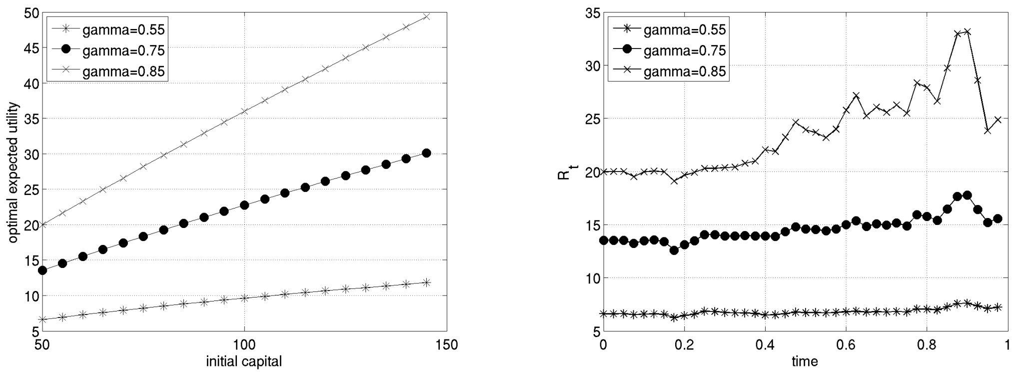
$$\begin{aligned} V_t &= 1 - \frac{q}{2} \int_t^T \frac{(\varphi_u + \lambda_u V_u)^2}{V_u} \sigma^2 S_u^2 \mathrm{d}u - \int_t^T \sigma S_u \varphi_u \mathrm{d}W_u \\ &= 1 - \frac{q}{2} \int_t^T \frac{(\tilde{\varphi}_u + \frac{\mu}{\sigma} V_u)^2}{V_u} \mathrm{d}u - \int_t^T \tilde{\varphi}_u \mathrm{d}W_u, \end{aligned}$$

where $q = \gamma/(1-\gamma)$ and $\tilde{\varphi}_t = \sigma S_t \varphi_t$. By the proof of Proposition (3.1.2), this BSDE transforms into a linear BSDE, hence it is amenable to any numerical scheme available for Lipschitz BSDEs. We consider a spot $S_0 = 100$ and take $F = (K - S_T)^+$ as an at-the-money European put option with a strike price of $K = 100$. Hence, the constant $c > 0$ from (3.20) is the Black-Scholes price of the put, and η is its corresponding delta hedge. For numerical simulations, we assume $T = 1$ and use the discretized Picard iteration algorithm from Bender and Denk [13] with a regression basis of 6 monomials, 40 equidistant time points in $[0, 1]$ and $100,000$ Monte Carlo simulation paths. The

Picard iteration terminates if two successive time zero values of the value function are below 10^{-7}. We see in figure 3.1(a) that higher values of γ lead to higher optimal expected utility values, i.e. the less risk averse the investor is, the more she can expect. Figure 3.1(b) depicts the pathwise supermartingale property of the BSDE value process V: starting at a high time value, it decreases to its target terminal value $\xi = 1$ as time evolves. Figure 3.2 depicts a sample path of the optimal strategy, illustrating that risk seeking investors in general must exhibit higher market interaction to achieve optimally.

3.3.2 Example for the multiplicative power utility case

This section provides a numerical example for the multiplicative optimization problem (3.29). We adopt the one-dimensional Black-Scholes model from Section 3.3.1. We



(a) Optimal expected utility $V(0,x)$ in dependence of the initial capital x.

(b) Pathwise martingale property of R_t at initial capital $x = 50$.

Figure 3.3: Optimal expected utility and pathwise martingale property of R_t.

assume that the liability F represents a simple two-level tax policy of the government which requires the investor to pay a high tax rate if $S_T/S_0 > 1$, i.e. when the stock outperforms the spot at maturity T, and to pay a low tax rate if $S_T/S_0 \le 1$, hence if the stock underperforms. More precisely, we assume

$$F = (1 - 0.48)1_{\{S_T > S_0\}} + (1 - 0.3)1_{\{S_T \le S_0\}}, \tag{3.35}$$

i.e. the investor pays 48% tax during a good run and 30% tax during a bad run of the stock. The stock evolves according to the Black-Scholes model with the parameters

$$\frac{\mathrm{d}S_t}{S_t} = \mu \mathrm{d}t + \sigma \mathrm{d}W_t = 0.05\mathrm{d}t + 0.29\mathrm{d}W_t,$$

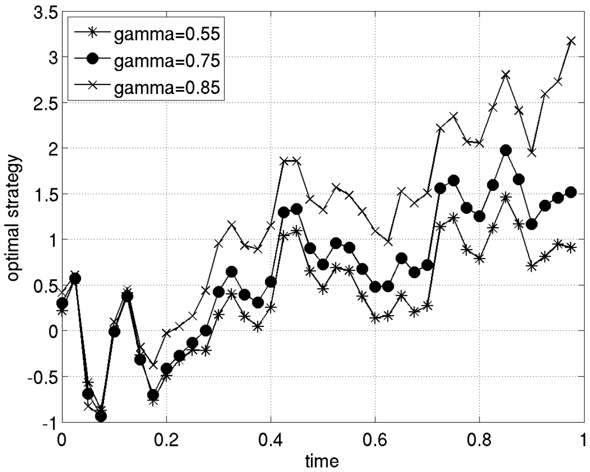


Figure 3.4: Sample paths of the dynamic evolution of the investment process at different risk aversion levels of γ.

with maturity $T = 1$. For the sake of convenience, we consider the optimization problem (3.29) without trading constraints. In this case, the BSDE (3.31) reads

$$\begin{aligned} Y_t &= \gamma \log(F) + \int_t^T \left(\frac{\gamma}{2(1-\gamma)} |(Z_u + \lambda_u)\sigma S_u|^2 + \frac{1}{2} |\sigma S_u Z_u|^2 \right) \mathrm{d}u - \int_t^T \sigma S_u Z_u \mathrm{d}W_u \\ &= \gamma \log(F) + \int_t^T \left(\frac{\gamma}{2(1-\gamma)} \big|(\tilde{Z}_u + \frac{\mu}{\sigma})\big|^2 + \frac{1}{2} |\tilde{Z}_u|^2 \right) \mathrm{d}u - \int_t^T \tilde{Z}_u \mathrm{d}W_u. \end{aligned}$$

According to Theorem 7 in Imkeller et al. [69] (see also Theorem 4.2.1 in Chapter 4), this type of quadratic BSDEs allows an exponential transformation into a Lipschitz BSDE which then can be solved numerically, e.g. by the discretized Picard iteration scheme from Bender and Denk [13]. We choose 40 equidistant time points in $[0, 1]$ and $100,000$ Monte Carlo paths with 7 monomials as regression basis. The Picard iteration terminates when the difference of two successive time zero values are below 10^{-7}. In Figure 3.3(a) we see that lower risk aversion leads to higher optimal expected utilities. Figure 3.3(b) shows the pathwise martingale property of the process $R^{x,\rho}$. The depiction of the optimal investment strategies in Figure 3.4 reveals that the more risk tolerance one admits the more trading activity one has to exhibit.

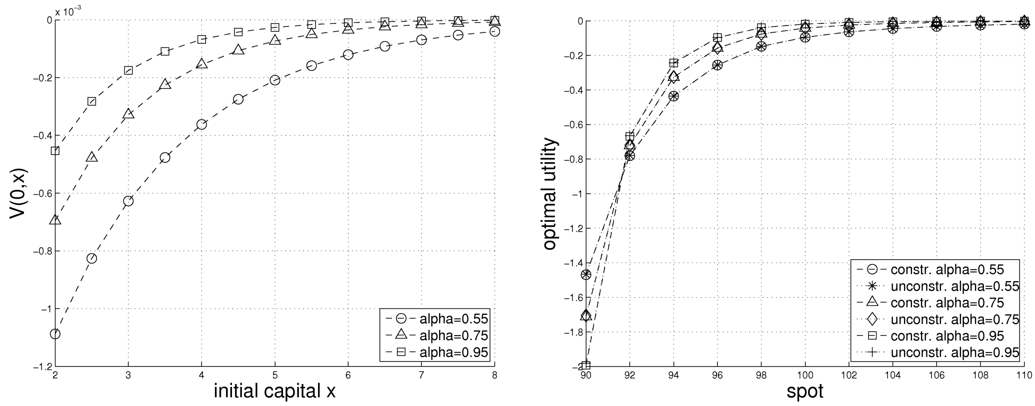
3.3.3 Example on the exponential utility case

We complement the numerical study by the case of the exponential utility function. In the context of Hu et al. [62], the exponential utility maximization problem boils down to solving a linear BSDE if there are no constraints on the investment strategy. However, if constraints come into play, one has to a solve a BSDE with a quadratic growth generator of the form (3.27) which for general constraint sets C can become intricate. If nevertheless the constraints set C offers regular enough features, the BSDE from Lemma 3.2.3 becomes numerically tractable. Such is the case if, for instance, C equals the set of all integers, meaning that investments have to be integer valued. If we accept this as a stylized fact, the quadratic term of the generator from equation (3.27) becomes bounded, hence Lipschitz continuous. This way we obtain a standard Lipschitz BSDE accessible to standard schemes e.g. those by Bouchard and Touzi [24], Bender and Denk [13].

Since Hu et al. [62] give a complete analysis of exponential utility maximization, we base the numerical simulations on the framework of Hu et al. [62] by specifying the constraint set to be the set of integers. More precisely, we consider a one-dimensional Black-Scholes market setup with zero interest rate bank account and a stock

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t, \;\; S_0 > 0.$$

The objective is to maximize the expected utility (3.25) given the liability of a European put option $F = (K - S_T)^+$. Note that since the driver is Lipschitz continuous, any square integrable liability F could be considered. The scheme of choice is the Picard iteration

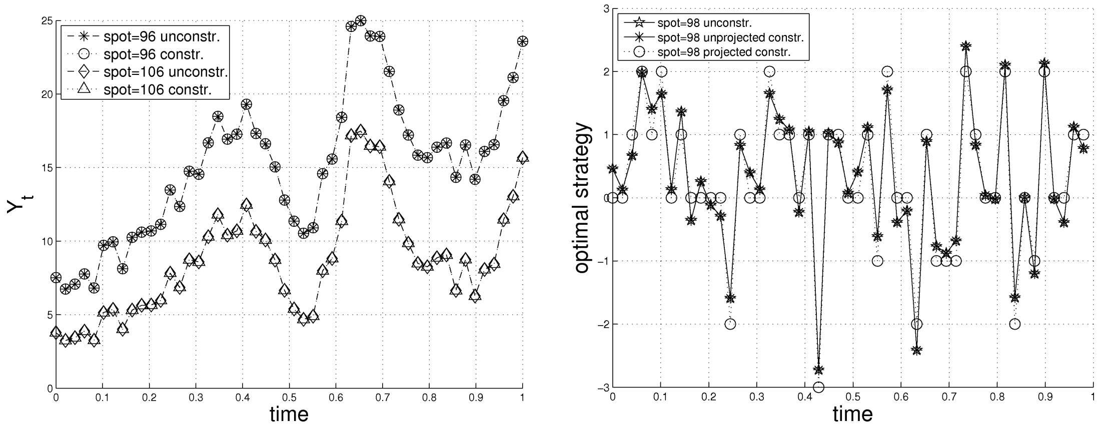


(a) Optimal expected utility $V(0,x)$ in dependence of the initial capital x for $S_0 = 100$.

(b) Optimal expected utility at initial capital $x = 10$ for different risk aversion parameters.

Figure 3.5: Optimal expected utilities.

scheme from Bender and Denk [13]. We simulate 50,000 paths with a regression basis of 6 monomials. The iteration stops if two successive time zero values of the value process Y lie within an error tolerance of 10^{-7}. The market parameters are $\mu = 0.05$, $\sigma = 0.2$, $T = 1$ and $K = 100$ and the spot S_0 varies between 90 and 110. Figure 3.5(a) depicts the optimal expected utility for different risk aversion levels in the presence of an at-the-money put option: higher risk aversion levels result in higher expected utilities. Figure 3.5(b) shows the expected utilities for various levels of risk aversion in dependence of the spot price S_0. It reveals the feature that the difference between the linear (unconstrained) case and the non-linear (constrained) case is negligible. This should not come as a surprise since the linear and the nonlinear generator from equation (3.24) are *close* to each other, meaning that they differ only by a quantity strictly less than 1. Effectively, the non-linear BSDE is "almost" linear. This property of being close to each other is underlined by Figure



(a) Plot of paths the value process Y for the linear (unconstrained) and nonlinear (constrained) case.

(b) Path of the optimal investment process and its projection onto the set of integers.

Figure 3.6: Path plots of the BSDE value process Y and the optimal strategies.

3.6(a): the value process of the linear (unconstrained) BSDE is hardly distinguishable from the value process of the non-linear (constrained) BSDE. Figure 3.6(b) depicts the optimal strategy as the projection onto integers. This figure visualizes the explanation for being close: since the difference between the projected and unprojected control process is small (and even becomes smaller by taking it to the square), the linear BSDE is an accurate proxy for the non-linear BSDE. Moreover, we see in Figure 3.6(b) that the control processes of the linear (unconstrained) and the non-linear (constrained) hardly differ.

4 A Cole-Hopf transformation for quadratic FBSDEs

In this chapter, we present a method for numerically approximating decoupled forward-backward stochastic differential equations with drivers of quadratic growth (qgFBSDEs). In the previous chapters, we have seen how qgFBSDEs can be used to solve utility maximization problems. In this chapter, we depict another illustration for the significance of qgFBSDEs: we discuss a problem of cross hedging of an insurance related financial derivative using correlated assets. For the convergence of numerical approximation schemes for such systems of stochastic equations, path regularity of the solution processes is imperative. We discuss a reduction method to FBSDEs with globally Lipschitz continuous drivers by using a Cole-Hopf type transformation. We then complement our method by numerical simulations for the pricing and hedging of simple insurance derivatives.

4.1 Preliminaries

We work on the canonical Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$ on which a d-dimensional Wiener process $W = (W^1, \cdots, W^d)$ restricted to the time interval $[0,T]$ is defined. Here, $T > 0$ is a fixed constant. We denote by $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ the natural filtration generated by W and enlarged by its $\mathbb{P}$-zero sets. Let $p \geq 2$ and $m \in \mathbb{N}$, and let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{F})$. We denote by $\mathbb{E}^{\mathbb{Q}}$ the expectation with respect to $\mathbb{Q}$, and we omit the superscript if $\mathbb{Q}$ coincides with the canonical measure $\mathbb{P}$. We denote the stochastic integral process of an adapted process Z with respect to the Wiener process by $Z * W = \int_0^\cdot Z_s \mathrm{d}W_s$. For vectors $x = (x^1, \cdots, x^m)$ in $\mathbb{R}^m$ we denote $|x| = (\sum_{i=1}^m (x^i)^2)^{\frac{1}{2}}$. In our analysis the following normed vector spaces will play a role. We denote by

- $L^p(\mathbb{R}^m; \mathbb{Q})$ the space of $\mathcal{F}_T$-measurable random variables $X : \Omega \to \mathbb{R}^m$, normed by $\|X\|_{L^p} = \mathbb{E}^{\mathbb{Q}}[\,|X|^p]^{\frac{1}{p}}$; L^∞ the space of bounded random variables;
- $\mathcal{S}^p(\mathbb{R}^m)$ the space of adapted processes $(Y_t)_{t\in[0,T]}$ with values in $\mathbb{R}^m$ normed by $\|Y\|_{\mathcal{S}^p} = \mathbb{E}[\big(\sup_{t\in[0,T]} |Y_t|\big)^p]^{\frac{1}{p}}$; $\mathcal{S}^\infty(\mathbb{R}^m)$ the space of bounded measurable processes;
- $\mathcal{H}^p(\mathbb{R}^m, \mathbb{Q})$ the space of progressively measurable processes $(Z_t)_{t\in[0,T]}$ with values in $\mathbb{R}^m$ normed by $\|Z\|_{\mathcal{H}^p} = \mathbb{E}^{\mathbb{Q}}[\Big(\int_0^T |Z_s|^2 \mathrm{d}s\Big)^{p/2}]^{\frac{1}{p}}$;
- $BMO(\mathcal{F}, \mathbb{Q})$ or $BMO_2(\mathcal{F}, \mathbb{Q})$ the space of square integrable $\mathcal{F}$-martingales Φ with

$\Phi_0 = 0$ and equipped with the norm

$$\|\Phi\|^2_{BMO(\mathcal{F},\mathbb{Q})} = \sup_{\tau} \Big\| \mathbb{E}^{\mathbb{Q}}\big[\langle\Phi\rangle_T - \langle\Phi\rangle_\tau | \mathcal{F}_\tau\big]\Big\|_\infty < \infty,$$

where the supremum is taken over all stopping times τ with values in $[0,T]$. In cases where the measure $\mathbb{Q}$ resp. the filtration $\mathcal{F}$ is clear from the context, we omit $\mathbb{Q}$ or $\mathcal{F}$ and simply write $BMO(\mathbb{Q})$ resp. $BMO(\mathcal{F})$ etc.

In case when there is no ambiguity about m or $\mathbb{Q}$, we omit the reference to $\mathbb{R}^m$ or $\mathbb{Q}$ and simply write $\mathcal{S}^\infty$ or $\mathcal{H}^p$ etc.

We investigate systems of forward diffusions coupled with backward stochastic differential equations of quadratic growth in the control variable (qgFBSDE for short), i.e. given $x \in \mathbb{R}^m$, $t \in [0,T]$, and four continuous measurable functions b, σ, g and f we analyze systems of the form

$$X_t^x = x + \int_0^t b(s, X_s^x)\mathrm{d}s + \int_0^t \sigma(s, X_s^x)\mathrm{d}W_s, \tag{4.1}$$

$$Y_t^x = g(X_T^x) + \int_t^T f(s, X_s^x, Y_s^x, Z_s^x)\mathrm{d}s - \int_t^T Z_s^x \mathrm{d}W_s. \tag{4.2}$$

In case when there is no ambiguity about the initial state x of the forward system, we suppress the superscript x and simply denote X, Y, Z for the components of the solution. For the coefficients of this system we make the following assumptions:

(H0) There exists a positive constant K such that $b, \sigma_i : [0,T]\times\mathbb{R}^m \to \mathbb{R}^m, 1 \le i \le d$, are uniformly Lipschitz continuous with Lipschitz constant K, and $b(\cdot,0)$ and $\sigma_i(\cdot,0), 1 \le i \le d$, are bounded by K.

There exists a constant $M > 0$ such that $g : \mathbb{R}^m \to \mathbb{R}$ is bounded by M, $f : [0,T]\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ is measurable and continuous in (x,y,z) and for $(t,x) \in [0,T]\times\mathbb{R}^m$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^d$ we have

$$|f(t,x,y,z)| \le M(1 + |y| + |z|^2),$$
$$|f(t,x,y,z) - f(t,x,y',z')| \le M\Big\{|y-y'| + (1 + |z| + |z'|)|z-z'|\Big\}.$$

We have the following standard existence and existence result.

Theorem 4.1.1. *Under **(H0)**, the system (4.1), (4.2) has a unique solution $(X,Y,Z) \in \mathcal{S}^2\times\mathcal{S}^\infty\times\mathcal{H}^2$. The respective norms of Y and Z can be dominated from above by constants depending only on T and M. Furthermore,*

$$Z * W = \int_0^{\cdot} Z_s \mathrm{d}W_s \in BMO(\mathbb{P}),$$

and hence for all $p \ge 2$ one has $Z \in \mathcal{H}^p$.

4.2 The exponential transformation method

In this section we discuss an approach to smoothness of solutions in a particular situation that the driver satisfies a structure condition. This approach makes use of an exponential transformation which resembles the Cole-Hopf transformation known from PDE theory. This mapping consists of considering e^Y and has the feature that it eliminates quadratic terms in the control variable of the form $\gamma|z|^2$. The price one has to pay for this approach is a possibly missing global Lipschitz condition in the variable y for the modified driver. It is therefore not clear if the new BSDE is amenable to the usual numerical discretization techniques. We give sufficient conditions for the transformed driver to satisfy a global Lipschitz condition. In this simpler setting our techniques allow an easier access to smoothness results for the solutions of the transformed BSDE. However, since the exponential transformation is one-to-one, the regularity results carry over to the original qgFBSDE.

Under **(H0)**, we consider the transformation $P = e^{\gamma Y}$ and $Q = \gamma P Z$ where $\gamma \in \mathbb{R}$ is a constant which comes from the quadratic growth driver in (4.6). It transforms the qgBSDE (4.2) with driver f into the new BSDE

$$P_t = e^{\gamma g(X_T)} + \int_t^T \Big[\gamma P_s f\Big(s, X_s, \frac{\log P_s}{\gamma}, \frac{Q_s}{\gamma P_s}\Big) - \frac{1}{2}\frac{|Q|_s^2}{P_s}\Big]\mathrm{d}s - \int_t^T Q_s \mathrm{d}W_s, \quad t \in [0,T]. \tag{4.3}$$

Combining (4.3) with the SDE (4.1), we see that for any $p \geq 2$ a unique solution $(X, P, Q) \in \mathcal{S}^p \times \mathcal{S}^\infty \times \mathcal{H}^p$ of (4.1) and (4.3) exists. The properties of this triplet follow from the properties of the solution (X, Y, Z) of the original qgFBSDE (4.1) and (4.2). To be more precise, since Y is bounded, P becomes bounded away from 0. This allows to deduce from the BMO martingale property of $Z * W$ that $Q * W$ is a BMO martingale.

For the rest of this section we denote by the range of P by $\mathcal{K}$, a compact subset of $(\delta, +\infty)$ for some constant $\delta > 0$. From now on, we work under the following hypothesis.

(H0*) Assume that **(H0)** holds. For $\gamma \in \mathbb{R}$ let $f : [0,T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be of the form

$$f(t,x,y,z) = l(t,x,y) + a(t,z) + \frac{\gamma}{2}|z|^2,$$

where l and a are measurable, l is uniformly Lipschitz continuous in x and y, a is uniformly Lipschitz continuous and homogeneous in z, i.e. for $c \in \mathbb{R}, (s,z) \in [0,T] \times \mathbb{R}^d$ we have $a(s, cz) = ca(s,z)$ and l and a are continuous in t.

The structure of the driver in (4.3) indicates that after transforming drivers satisfying **(H0*)**, we have good chances to deal with a Lipschitz continuous one. This is because assumption **(H0*)** allows to simplify the BSDE obtained from the exponential transformation to

$$P_t = e^{\gamma g(X_T)} + \int_t^T F(s, X_s, P_s, Q_s)\mathrm{d}s - \int_t^T Q_s \mathrm{d}W_s, \quad t \in [0,T], \tag{4.4}$$

where the driver is defined by

$$\begin{aligned} F : [0,T] \times \mathbb{R}^m \times \mathcal{K} \times \mathbb{R}^d &\to \mathbb{R}, \\ (s,x,p,q) &\mapsto \gamma p\, l\big(s,x,\frac{\log p}{\gamma}\big) + \gamma p\, a\big(s,\frac{q}{\gamma p}\big). \end{aligned} \tag{4.5}$$

Thanks to the homogeneity assumption on a our driver simplifies further. Indeed, we have for $(s,x,p,q) \in [0,T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d$

$$F(s,x,p,q) = \gamma p\, l\big(s,x,\frac{\log p}{\gamma}\big) + a\big(s,q\big). \tag{4.6}$$

The terminal condition of the transformed BSDE retains boundedness because the boundedness of g is inherited by $\exp(\gamma g)$. Furthermore, if g is uniformly Lipschitz, then again by the boundedness of g, the function $e^{\gamma g}$ is also uniformly Lipschitz.

Let us next discuss the properties of the driver (4.5) of the transformed BSDE. We recall that since l and a are Lipschitz continuous, there is a constant $C > 0$ such that for all $(s,x,p,q) \in [0,T] \times \mathbb{R}^m \times \mathcal{K} \times \mathbb{R}^d$ we have

$$\begin{aligned} |F(s,x,p,q)| &\le \Big|\gamma p\, l\big(s,x,\frac{\log p}{\gamma}\big) + a\big(s,q\big)\Big| \\ &\le C|p|\big(1 + |x| + |\log p| + |q|\big) \le C\big(1 + |x| + |p| + |q|\big). \end{aligned}$$

This means that F is of linear growth in x,p and q. To verify Lipschitz continuity properties of F in its variables x,p and q, by (4.6) and the Lipschitz continuity assumptions on a, it remains to verify that

$$(x,p) \mapsto \gamma p\, l(s,x,\frac{\log p}{\gamma})$$

is Lipschitz continuous in x and p, with a Lipschitz constant independent of $s \in [0,T]$. As for x, this is an immediate consequence of the Lipschitz continuity of l in x. For p we have to recall that p is restricted to a compact set $\mathcal{K} \subset \mathbb{R}_+$ not containing 0. This allows to invoke the Lipschitz continuiyt of l in y. Hence, F is globally Lipschitz continuous in its variables x,p and q. We summarize these observations.

Theorem 4.2.1. *Let $f : [0,T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a measurable function, continuous on $\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d$ which satisfies **(H0*)**. Then, F defined by (4.5) is a uniformly Lipschitz continuous function in the spatial variables.*

Now Theorem 4.2.1 opens a route to tackle a convergence proof of numerical schemes using the path regularity of the control component for drivers satisfying **(H0*)**. Since the transformed BSDE has a Lipschitz continuous driver, path regularity for the control component Q of the transformed BSDE will follow from Zhang's path regularity result (reporduced in Theorem 4.5.1), provided the driver is $\frac{1}{2}$-Hölder continuous in time. Of course, by the smoothness of the exponential transform, the control component Z of the original BSDE will inherit path regularity from Q. In what follows the triplets (X, Y, Z) and (X, P, Q) always refer to the solution of qgFBSDE (4.1), (4.2) and FBSDE (4.1), (4.4) respectively.

Theorem 4.2.2. *Assume **(H0*)**. Assume that* $[0,T] \times \mathbb{R}^m \times \mathcal{K} \times \mathbb{R}^d \ni (s,x,p,q) \mapsto F(s,x,p,q) \in \mathbb{R}$ *is uniformly Lipschitz continuous in* x, p *and* q *and is* $\frac{1}{2}$*-Hölder continuous in* s*. Moreover, suppose that the map* $g : \mathbb{R}^d \to \mathbb{R}$ *is bounded and globally Lipschitz continuous with Lipschitz constant* K*. Let* (X, Y, Z) *be the solution of qgFBSDE (4.1), (4.2). Let* $\varepsilon > 0$*. Then, there exists a constant* $\kappa > 0$ *such that for any partition* $\pi = \{t_0, \dots, t_N\}$ *with* $0 = t_0, T = t_N, t_0 < \cdots < t_N$*, of the interval* $[0,T]$ *of mesh size* $|\pi|$ *we have*

$$\max_{0\le i\le N-1}\Big\{\sup_{t\in[t_i,t_{i+1})}\mathbb{E}\Big[|Y_t - Y_{t_i}|^2\Big]\Big\} \le \kappa|\pi| \quad \text{and} \quad \sum_{i=0}^{N-1}\mathbb{E}\Big[\int_{t_i}^{t_{i+1}}|Z_s - \bar{Z}^{\pi}_{t_i}|^2\mathrm{d}s\Big] \le \kappa|\pi|^{1-\varepsilon}.$$

Moreover, if the functions b *and* σ *are continuously differentiable in* $x \in \mathbb{R}^m$*, then the mapping* $[0,T] \ni t \mapsto Z_t$ *is a.s. continuous.*

Proof. Throughout this proof $C > 0$ denotes a positive constant which may vary from line to line. Let (X, P, Q) be the solution of (4.1) and (4.4), where P takes its values in $\mathcal{K}$ and $Q * W$ is a BMO martingale. By Zhang's path regularity result, reproduced in Theorem 4.5.1, there exists $C > 0$ such that for any partition $\pi = \{t_0, \dots, t_N\}$ of $[0,T]$ with mesh size $|\pi|$

$$\max_{0\le i\le N-1}\Big\{\sup_{t\in[t_i,t_{i+1})}\mathbb{E}\Big[|P_t - P_{t_i}|^2\Big]\Big\} + \sum_{i=0}^{N-1}\mathbb{E}\Big[\int_{t_i}^{t_{i+1}}|Q_s - \bar{Q}^{\pi}_{t_i}|^2\mathrm{d}s\Big] \le C|\pi|.$$

Since P takes its values in the compact set $\mathcal{K} \subset \mathbb{R}_+$ not containing 0 there exists a constant C such that for any $0 \le i \le N-1, t \in [t_i, t_{i+1})$

$$|Y_t - Y_{t_i}| \le C|\log P_t - \log P_{t_i}| \le C|P_t - P_{t_i}|.$$

Using the two above inequalities we have

$$\max_{0\le i\le N-1}\Big\{\sup_{t\in[t_i,t_{i+1})}\mathbb{E}\Big[|Y_t - Y_{t_i}|^2\Big]\Big\} \le C\max_{0\le i\le N-1}\Big\{\sup_{t\in[t_i,t_{i+1})}\mathbb{E}\Big[|P_t - P_{t_i}|^2\Big]\Big\} \le C|\pi|.$$

This proves the first inequality. For the second one, note that by definition for $0 \le i \le N-1$ and $t \in [t_i, t_{i+1})$ we have

$$\begin{aligned}|Z_t - \bar{Z}_{t_i}| \le |Z_t - Z_{t_i}| &\le \frac{1}{\gamma}\Big\{\Big|\frac{Q_t}{P_t} - \frac{Q_t}{P_{t_i}}\Big| + \Big|\frac{Q_t}{P_{t_i}} - \frac{Q_{t_i}}{P_{t_i}}\Big|\Big\} \le \frac{1}{\gamma}\Big\{|Q_t|\Big|\frac{1}{P_t} - \frac{1}{P_{t_i}}\Big| + \frac{1}{|P_{t_i}|}|Q_t - Q_{t_i}|\Big\}\\ &\le C\Big\{|Q_t|\;|P_t - P_{t_i}| + |Q_t - Q_{t_i}|\Big\}.\end{aligned}$$

We therefore have for $0 \le i \le N-1$

$$\mathbb{E}\Big[\int_{t_i}^{t_{i+1}} |Z_s - \bar{Z}^\pi_{t_i}|^2 \mathrm{d}s\Big] \le \mathbb{E}\Big[\int_{t_i}^{t_{i+1}} |Z_s - Z_{t_i}|^2 \mathrm{d}s\Big]$$
$$\le 2C\Big\{\mathbb{E}\Big[\sup_{t\in[t_i,t_{i+1})} |P_t - P_{t_i}|^2 \int_{t_i}^{t_{i+1}} |Q_s|^2 \mathrm{d}s\Big] + \mathbb{E}\Big[\int_{t_i}^{t_{i+1}} |Q_t - Q_{t_i}|^2 \mathrm{d}s\Big]\Big\}.$$

Since $Q \in \mathcal{H}^p$ for all $p \ge 2$, for any two real numbers $\alpha, \beta \in (1,\infty)$ satisfying $1/\alpha + 1/\beta = 1$ we can apply Hölder's inequality on the right-hand side of the inequality, and then Theorem 4.5.1 to the term containing P to get

$$\mathbb{E}\Big[\int_{t_i}^{t_{i+1}} |Z_s - \bar{Z}^\pi_{t_i}|^2 \mathrm{d}s\Big] \le C\Big\{\mathbb{E}\Big[\sup_{t\in[t_i,t_{i+1})} |P_t - P_{t_i}|^{2\alpha}\Big]^{\frac{1}{\alpha}} \mathbb{E}\Big[\Big(\int_{t_i}^{t_{i+1}} |Q_s|^2 \mathrm{d}s\Big)^\beta\Big]^{\frac{1}{\beta}} + |\pi|\Big\}$$
$$\le C\Big\{\mathbb{E}\Big[\sup_{t\in[t_i,t_{i+1})} |P_t - P_{t_i}|^2\Big]^{\frac{1}{\alpha}} + |\pi|\Big\} \le C\Big\{|\pi|^{\frac{1}{\alpha}} + |\pi|\Big\}.$$

for every $0 \le i \le N-1$. Note that the constant $C > 0$ is independent of i. Choosing $\alpha = \frac{1}{1-\varepsilon}$ and $\kappa = C$ completes the estimate of the claim.

To prove that Z admits almost surely a continuous version, it is enough to remark that the assumptions imply the conditions of Corollary 5.6 from Ma and Zhang [87]. This result yields that Q is a.s. continuous on $[0,T]$. Since P is continuous and bounded away from zero we conclude from the equation $\gamma P Z = Q$ that Z is a.s. continuous. □

4.3 Pricing and hedging with correlated assets

In this section, we consider the indifference pricing of a contingent claim by means of FBSDEs. In incomplete markets, one established pricing paradigm is indifference pricing which is closely related to utility maximization problems studied in the previous chapters. Upon choosing a risk preference, investors evaluate contingent claims by replicating according to an investment strategy that yields the most favorable utility value. Interplays and connections between the pricing of contingent claims on non-tradable underlyings and the theory of qgFBSDE were studied, among others, by Ankirchner et al. [6], Morlais [96], Imkeller et al. [68] and Frei [53]. Adopting the setup from these works, we consider the problem of numerically evaluating contingent claims using non-tradable underlyings.

The following toy market setup is considered in Section 4 in Frei [53]. Assume $d = 2$, i.e. $W = (W^1, W^2)$ which we use to define a third Brownian motion W^3,

$$W^3_s := \int_0^s \rho \mathrm{d}W^1_u + \int_0^s \sqrt{1-\rho^2}\mathrm{d}W^2_u, \quad 0 \le s \le T.$$

Obviously, W^3 is correlated to W^1 with the correlation coefficient $\rho \in [-1,1]$. Contingent claims are assumed to be tied to a one-dimensional non-tradable asset (e.g. a stock index) that is subject to

$$\mathrm{d}R_t = \mu(t, R_t)\mathrm{d}t + \sigma(t, R_t)\mathrm{d}W_t^1,\ R_0 = r_0 > 0, \tag{4.7}$$

where $\mu, \sigma : [0,T] \times \mathbb{R} \to \mathbb{R}$ are deterministic measurable and uniformly Lipschitz continuous functions, uniformly of (at most) linear growth in their state variable. The securities market is governed by a risk free bank account yielding zero interest and one correlated risky asset whose dynamics (with respect to the zero interest bank account *numéraire*) is governed by

$$\frac{\mathrm{d}S_s}{S_s} = \alpha(s, R_s)\mathrm{d}s + \beta(s, R_s)\mathrm{d}W_s^3,\ S_0 = s_0 > 0. \tag{4.8}$$

In compliance with Ankirchner et al. [6], we assume that $\alpha, \beta : [0,T] \times \mathbb{R} \to \mathbb{R}$ are bounded and measurable functions, and furthermore $\beta^2(t,r) \geq \varepsilon > 0$ holds uniformly for some fixed $\varepsilon > 0$. Next, we set

$$\theta(s,r) := \frac{\alpha(s,r)}{\beta(s,r)},\quad (s,r) \in [0,T] \times \mathbb{R},$$

and note that the conditions on α and β imply that θ is uniformly bounded.

We focus on European style contingent claims, i.e. payoff profiles resuming the form $F(R_T)$ where we assume, in accordance with Ankirchner et al. [6], that $F : \mathbb{R} \to \mathbb{R}$ is measurable and bounded. Moreover, the investor's risk assessment is given by the exponential utility function, so given a constant risk attitude parameter $\eta > 0$, the investor's utility function is

$$U(x) = -e^{-\eta x},\ x \in \mathbb{R}.$$

An admissible investment strategy is defined to be a real valued, measurable predictable process λ such that $\int_0^T \lambda_u^2 \mathrm{d}u < \infty$ holds $\mathbb{P}$-almost surely and such that the family

$$\left\{ \left| U\Big(\int_0^\tau \lambda_u \frac{\mathrm{d}S_u}{S_u}\Big) \right| = e^{-\eta \int_0^\tau \lambda_u \frac{\mathrm{d}S_u}{S_u}} : \tau \text{ stopping time with values in } [0,T] \right\} \tag{4.9}$$

is uniformly integrable. The set of all admissible investment strategies is denoted by $\mathcal{A}$. In the following, let $t \in [0,T]$ denote a fixed time. Then the set of all admissible investment strategies living on the time interval $[t,T]$ is defined analogously and we denote it by $\mathcal{A}_t$. Let v_t denote the investor's initial endowment at time t, that is, v_t is an $\mathcal{F}_t$-measurable random variable. The gain of the investor at time $s \in [t,T]$, denoted by G_s, is subject to trading according to the investment strategy λ and therefore given by

$$\mathrm{d}G_s^\lambda = \lambda_s \frac{\mathrm{d}S_s}{S_s},\ G_t^\lambda = 0.$$

The evolution of the investor's portfolio over the time interval $[t,T]$ consists of her initial endowment v_t, her gains (or losses) via her strategy λ and holding one share of the contingent claim $F(R_T)$. Her objective is to find an investment strategy such that her time-t utility is maximized, i.e. her maximization problem is given by

$$\begin{aligned} V_t^F(v_t) &:= \operatorname{esssup}\left\{\mathbb{E}\left[U(v_t + G_T^\lambda + F(R_T))\middle|\mathcal{F}_t\right] : \lambda \in \mathcal{A}_t\right\} \\ &= \exp\{-\eta v_t\} \operatorname{esssup}\left\{\mathbb{E}\left[U(G_T^\lambda + F(R_T))\middle|\mathcal{F}_t\right] : \lambda \in \mathcal{A}_t\right\} \end{aligned} \tag{4.10}$$

For the sake of notational convenience, we write

$$V_t^F := V_t^F(0) = \operatorname{esssup}\left\{\mathbb{E}\left[U(G_T^\lambda + F(R_T))\middle|\mathcal{F}_t\right] : \lambda \in \mathcal{A}_t\right\}. \tag{4.11}$$

Now the indifference price for $F(R_T)$ is given by an $\mathcal{F}_t$-measurable random variable p_t which satisfies the identity

$$V_t^0(v_t) = V_t^F(v_t - p_t),$$

where $V_t^0(v_t)$ denotes the time-t utility with initial endowment v_t and with $F = 0$ (see also Section 2 of Ankirchner et al. [6] and Section 3 of Frei [53]). According to this identity, the investor is indifferent about a portfolio with initial endowment v_t without receiving one quantity of the contingent claim $F(R_T)$ and a portfolio with initial endowment $v_t - p_t$ with receiving the contingent claim. Hence p_t is interpreted as the time-t indifference price of the contingent claim $F(R_T)$. By the equality $V_t^F(v_t) = \exp\{-\eta v_t\} V_t^F$, it follows that

$$p_t = \frac{1}{\eta} \log \frac{V_t^0}{V_t^F}, \tag{4.12}$$

which means that the indifference price does not depend on the initial endowment v_t. Since the time-t indifference price (4.12) is fully characterized by V_t^0 and V_t^F, the focus now lies in the investigation of (4.10), (4.11). In fact, Ankirchner et al. [6] and Frei [53] have already pointed out that (4.11) can be characterized by means of a qgFBSDE. In accordance with Frei [53], let us denote by $(\mathcal{G}_u)_{0 \le u \le T}$ the filtration generated by W^1, completed by $\mathbb{P}$-null sets. The main ideas from Frei [53] for rephrasing (4.10) in terms of a qgBSDE are summarized in the following result.

Lemma 4.3.1. *The qgFBSDE*

$$Y_s = F(R_T) + \int_s^T f(u, R_u, Z_u)\mathrm{d}u - \int_s^T Z_u \mathrm{d}W_u^1, \quad s \in [0, T], \tag{4.13}$$

$$f(u, r, z) = \frac{\theta^2(u, r)}{2\eta} - z\rho\theta(u, r) - \frac{\eta}{2}\left(1 - \rho^2\right) z^2, \tag{4.14}$$

has a unique solution $(Y, Z) \in \mathcal{S}^\infty \times \mathcal{H}^2$ such that $V_t^F = -e^{-\eta Y_t}$ holds $\mathbb{P}$-almost surely.

Proof. Since $\theta(\cdot, r)$ is uniformly bounded and $\mathcal{G}$-predictable, the driver of (4.13) satisfies the conditions of Kobylanski [77]; thus (4.13) admits a unique solution $(Y, Z) \in \mathcal{S}^\infty \times \mathcal{H}^2$. Moreover, Mania and Schweizer [91] have shown that $Z * W^1$ is both a BMO($\mathcal{F}$)- and

BMO($\mathcal{G}$)-martingale, see also Ankirchner et al. [6, 4]. To prove the identity $V_t^F = -e^{-\eta Y_t}$, we notice that

$$e^{-\eta\left(G_T^\lambda + Y_T\right)} = e^{-\eta G_t^\lambda} e^{-\eta Y_t} e^{-\eta (Y_T - Y_t)} e^{-\eta\left(G_T^\lambda - G_t^\lambda\right)}$$
$$= e^{-\eta Y_t} e^{-\eta (Y_T - Y_t)} e^{-\eta\left(G_T^\lambda - G_t^\lambda\right)},$$

because $G_t^\lambda = 0$. We then have

$$\exp\{-\eta\,(Y_T - Y_t)\}\exp\{-\eta(G_T^\lambda - G_t^\lambda)\}$$
$$= \exp\left\{-\eta\left(\int_t^T Z_u \mathrm{d}W_u^1 + \int_t^T \lambda_u \beta(u, R_u)\mathrm{d}W_u^3 + \int_t^T \left[\lambda_u \alpha(u, R_u) - f(u, R_u, Z_u)\right]\mathrm{d}u\right)\right\}.$$

Denoting $\mathcal{E}_t^s(M) = \mathcal{E}(M)_s / \mathcal{E}(M)_t$ for $t \le s \le T$ where $\mathcal{E}(M)_s$ is the stochastic exponential of a given semimartingale M, we denote

$$K_u := \frac{1}{2}\Big(\eta\,(\rho Z_u + \beta(u, R_u)\lambda_u) - \theta_u\Big)^2, \quad t \le u \le T.$$

Then a straightforward calculation yields

$$\exp\Big\{-\eta\big(Y_T - Y_t\big)\Big\}\exp\Big\{-\eta\big(G_T^\lambda - G_t^\lambda\big)\Big\}$$
$$= \mathcal{E}_t^T\left(\int -\eta Z \mathrm{d}W^1 - \int \eta\lambda\beta(\cdot, R)\mathrm{d}W^3\right)\exp\left\{\int_t^T K_u \mathrm{d}u\right\}.$$

Since $\lambda\beta(\cdot, R) * W^3$ is a BMO-martingale, we can condition onto $\mathcal{F}_t$ and get

$$\mathbb{E}\left[e^{-\eta\left(G_T^\lambda + F(R_T)\right)} \;\middle|\; \mathcal{F}_t\right] = e^{-\eta Y_t} e^{\int_t^T K_u \mathrm{d}u} \ge e^{-\eta Y_t}. \tag{4.15}$$

By (4.9) and a localization argument, this inequality holds for every $\lambda \in \mathcal{A}_t$, and therefore we have $V_t^F \le -e^{-\eta Y_t}$. To prove equality, note that the inequality (4.15) becomes an equality for $\tilde{\lambda}_u = -\frac{\rho}{\beta(u,R_u)} Z_u + \frac{\theta(u,R_u)}{\eta\beta(u,R_u)}$ because this implies $K \equiv 0$. This in conjunction with the observation that

$$\exp\left\{-\eta\int_t^T \tilde{\lambda}_u \frac{\mathrm{d}S_u}{S_u}\right\} = \exp\left\{-\eta G_T^{\tilde{\lambda}}\right\} = \exp\left\{-\eta\left(G_T^{\tilde{\lambda}} - G_t^{\tilde{\lambda}}\right)\right\}$$
$$= \mathcal{E}_t^T\left(\int -\eta Z \mathrm{d}W^1 - \int \eta\tilde{\lambda}\beta(\cdot, R)\mathrm{d}W^3\right) \times \exp\left\{-\eta\,(Y_T - Y_t)\right\}$$

is the product of a bounded process and a true $\mathcal{F}$-martingale yields that condition (4.9) is satisfied. Hence $\tilde{\lambda} \in \mathcal{A}_t$ and we have shown $V_t^F = -e^{-\eta Y_t}$. □

From the proof of Lemma 4.3.1 we get the following result.

Corollary 4.3.1. *The investment strategy*

$$\tilde{\lambda}_s := -\frac{\rho}{\beta(s,R_s)} Z_s + \frac{\theta(s,R_s)}{\eta\beta(s,R_s)}, \quad t \le s \le T, \tag{4.16}$$

where Z is the control component of the solution to (4.13)*, belongs to $\mathcal{A}_t$ and satisfies*

$$\mathbb{E}\left[U(v_t + G^{\tilde{\lambda}_T} + F(R_T)\big|\mathcal{F}_t\right] = esssup\left\{\mathbb{E}\left[U(v_t + G_T^{\lambda} + F(R_T)\big|\mathcal{F}_t\right] : \lambda \in \mathcal{A}_t\right\} = V_t^F(v_t).$$

Example 4.3.1. *[Example 1.2 from Ankirchner et al. [6]] We give an example where qgBSDE can be used for pricing and hedging of non-traded assets. In 2008/09, oil prices saw a considerable price decline. In such a market environment, companies producing kerosene wish to partially cover their risk of such a depreciation. European put options are an established financial instrument to comply with this demand of risk covering. Since kerosene is not traded in a liquid market, derivative contracts on this underlying must be arranged on an over-the-counter basis. Knowing that the price of heating oil is highly correlated with the price of kerosene, the pricing and hedging of a European put option on kerosene can be done by a dynamic investment in (the liquid market of) heating oil. A numerical treatment of this pricing problem is presented in Section 4.4.*

4.4 Numerical experiments

In this section, we give a numerical treatment of the pricing problem from Example 4.3.1. Assume that the put option expires at $T = 1$. Let R and S denote the dynamics for the financial value of kerosene and heating oil respectively. In particular we assume both dynamics to be lognormal, i.e.

$$\begin{aligned}
\mathrm{d}R_t &= \mu(t,R_t)\mathrm{d}t + \sigma(t,R_t)\mathrm{d}W_t^1 = 0.12\,R_t\,\mathrm{d}t + 0.41\,R_t\,\mathrm{d}W_t^1, \\
\frac{\mathrm{d}S_t}{S_t} &= \alpha(t,R_t)\mathrm{d}t + \beta(t,R_t)\mathrm{d}W_t^3 = 0.1\,\mathrm{d}t + 0.35\,\mathrm{d}W_t^3,
\end{aligned}$$

and we assume the spot price for heating oil to be $s_0 = 173$ money units (e.g. US Dollar, Euro). Risk aversion is set at the level of $\eta = 0.3$. Figure 4.1 displays sample paths of the kerosene price with a spot price of $r_0 = 170$ and heating oil price at different correlation levels using the explicit solution formula for the geometric Brownian motion. We see that the higher the correlation, the better the approximation of the kerosene by heating oil becomes. We have seen that the valuation of the put option via utility maximization yields the pricing formula (4.12) which in conjunction with Lemma 4.3.1 becomes the difference of two solutions of a qgBSDE with the generator (4.14)

$$p_t = Y_t^F - Y_t^0, \quad 0 \le t \le T,$$

where $F(x) = (K - x)^+$ for some strike $K > 0$. For the numerical simulation of the qgFBSDE Y^F and Y^0, we apply the exponential transformation to both BSDEs (see Section 4.2) and then employ the algorithm by Bender and Denk [13] with $N = 100$

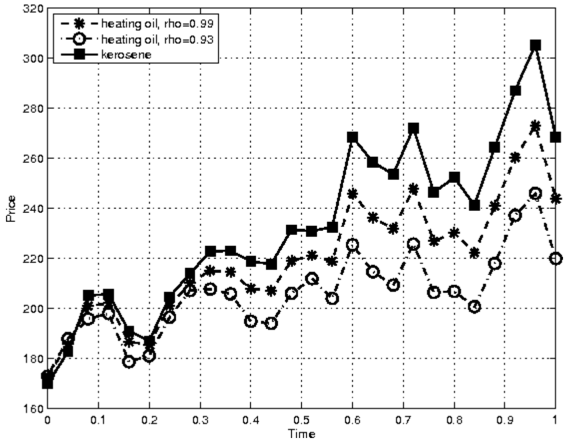
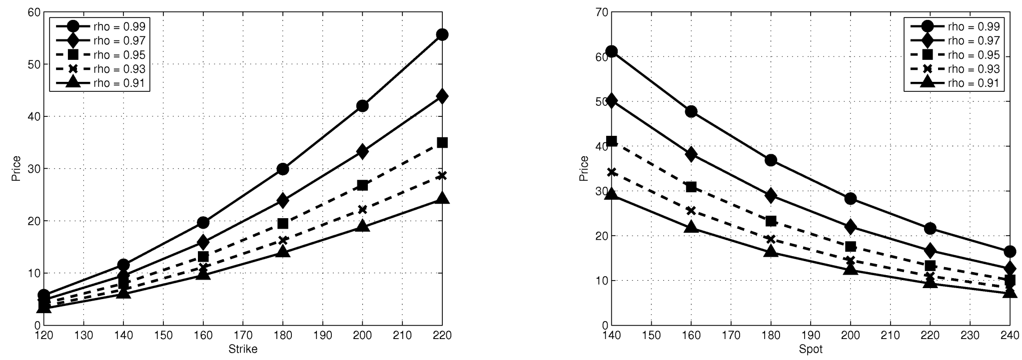


Figure 4.1: Price paths of the non-tradable asset kerosene and the correlated asset heating oil at different correlation levels. The spot of kerosene was set to $r_0 = 170$.

equidistant time points, a sample size of 70, 000 Monte Carlo paths and a regression basis consisting of five monomials and the payoff function. The Picard iteration stops as soon as the difference of two subsequent time zero values is less than 10^{-5}. Our numerical experiments reveal that 12 to 13 iterations are needed for solving one exponentially transformed qgFBSDE. Figures 4.2(a) and 4.2(b) depict the time zero price p_0 of the

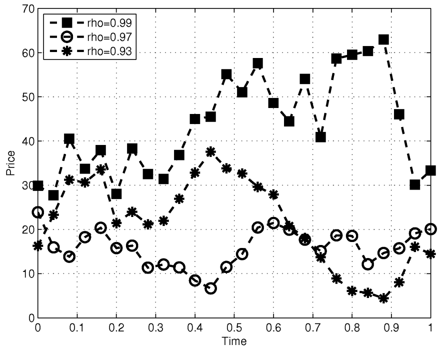


(a) Put option price in terms varying strikes at a fixed kerosene spot $r_0 = 170$.

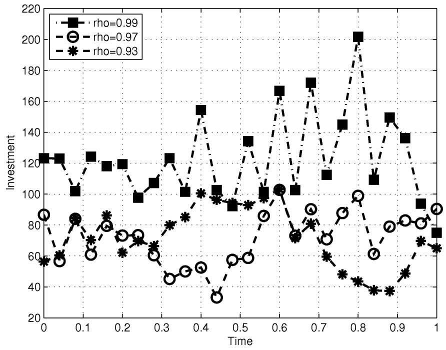
(b) Put option price in terms of varying kerosene spots at a fixed strike $K = 200$.

Figure 4.2: Values of the put option in terms of kerosene spot and strike for varying correlations. High correlations lead to high the prices for the contingent claim.

put option at different strike and kerosene spot levels. The lower the correlation, the lower the price. This is clear because lower correlations between heating oil and kerosene lead to higher non-hedgeable residual risk which diminishes the risk covering effect of the contingent claim and thus also its value. Figures 4.3(a) and 4.3(b) depict sample paths of



(a) Dynamics of the price process p_t for strike $K = 180$.



(b) Dynamics of the optimal investment strategy π_t for strike $K = 180$.

Figure 4.3: Paths of the price p_t and the optimal investment strategy π_t for varying correlation levels. High correlations entail greater market activity.

the dynamics for the price p_t and the optimal investment strategy π_t for the put option with strike $K = 180$ and kerosene spot $r_0 = 170$. The dynamics of the price process and the optimal investment strategy are intertwined: high fluctuations of the price process result in high fluctuations of the investment strategy and vice versa. In general we observe that replication on high correlation levels tends to entail greater market activity because kerosene price risks can then be well hedged by market transactions that move closely along the dynamics of heating oil. In contrast, replication on lower correlation levels leads to a higher amount of residual risk which is inaccessible for hedging and thus lower market activity is needed.

4.5 Repetition of path regularity for Lipschitz FBSDEs

We state a version of the L^2-regularity result for FBSDE satisfying a global Lipschitz condition. This result which was seen to be closely related to the convergence of numerical schemes for systems of FBSDE is due to Zhang [129]. For our FBSDE system (4.1), (4.2) we assume that b, σ, f, g are deterministic measurable functions that are Lipschitz continuous with respect to the spatial variables and $\frac{1}{2}$-Hölder continuous with respect to time. Furthermore we assume that σ satisfies

$$y^T \sigma(t,x)\sigma^T(t,x)y \geq c|y|^2, \quad x, y \in \mathbb{R}^m,\ t \in [0,T]. \tag{4.17}$$

for some constant $c > 0$. Then from El Karoui et al. [50] one easily obtains existence and uniqueness of a solution triple (X, Y, Z) of the FBSDE (4.1), (4.2) belonging to $\mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{H}^2$.

We now introduce a family of random variable $\bar{Z}^\pi_{t_i}$ which is needed for stating the path regularity result from Zhang [129]. Let $\pi = \{t_0, \cdots, t_N\}$ be a partition of $[0,T]$ with $N+1$ points and mesh size $|\pi|$. Let Z be the control component in the solution of the qgFBSDE (4.1), (4.2) under **(H1)** and define the family of random variables

$$\bar{Z}^{\pi}_{t_i} = \frac{1}{\Delta_i}\mathbb{E}\Big[\int_{t_i}^{t_{i+1}} Z_s \mathrm{d}s \Big| \mathcal{F}_{t_i}\Big], \quad t_i \in \pi \setminus \{t_N\}, \tag{4.18}$$

where $\Delta_i = t_{i+1} - t_i$. For $0 \leq i \leq N-1$ the random variable $\bar{Z}^{\pi}_{t_i}$ is the least squares $\mathcal{F}_{t_i}$-measurable approximation of Z in $\mathcal{H}^2([t_i, t_{i+1}])$, i.e.

$$\mathbb{E}\Big[\int_{t_i}^{t_{i+1}} |Z_s - \bar{Z}^{\pi}_{t_i}|^2 \mathrm{d}s\Big] = \inf_{\Lambda} \mathbb{E}\Big[\int_{t_i}^{t_{i+1}} |Z_s - \Lambda|^2 \mathrm{d}s\Big],$$

where Λ is allowed to vary in the space of all square integrable $\mathcal{F}_{t_i}$-measurable random variables.

Theorem 4.5.1 (Path regularity result of Zhang [129])**.** *Let $(X,Y,Z) \in \mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{H}^2$ be the solution of FBSDE (4.1), (4.2) in the setting described above. Then there exists a constant $C > 0$ such that for any partition $\pi = \{t_0, \cdots, t_N\}$ of the time interval $[0,T]$ with mesh size $|\pi|$ we have*

$$\begin{aligned} &\max_{0 \leq i \leq N-1} \Big\{ \sup_{t \in [t_i, t_{i+1})} \mathbb{E}\Big[|Y_t - Y_{t_i}|^2\Big]\Big\} \\ &\qquad + \sum_{i=0}^{N-1} \mathbb{E}\Big[\int_{t_i}^{t_{i+1}} |Z_s - \bar{Z}^{\pi}_{t_i}|^2 \mathrm{d}s\Big] + \sum_{i=0}^{N-1} \mathbb{E}\Big[\int_{t_i}^{t_{i+1}} |Z_s - Z_{t_i}|^2 \mathrm{d}s\Big] \leq C|\pi|. \end{aligned}$$

5 FBSDEs with time delayed generators

In this chapter we refine and extend the work of Delong and Imkeller [42, 43] concerning backward stochastic differential equations with time delayed generators (delay BSDE). We give moment and a priori estimates in general L^p-spaces and provide sufficient conditions for the solution of a delay BSDE to exist in L^p. We moreover introduce decoupled systems of SDEs and delay BSDEs (delay FBSDEs) and give sufficient conditions for their variational differentiability. We connect these variational derivatives to the Malliavin derivatives of delay FBSDEs using standard representation formulas. We conclude with several path regularity results, in particular we extend the classic L^2-path regularity result due to Zhang [129] (see Theorem 4.5.1 in Chapter 4) to delay FBSDEs.

5.1 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a standard d-dimensional Brownian motion W. For a fixed real number $T > 0$ we consider the filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ generated by W and augmented by all $\mathbb{P}$-null sets. The filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfies the usual conditions. Depending on whether we work on $\mathbb{R}^d$ or $\mathbb{R}^{m\times d}$, the Euclidean norm respectively the Hilbert-Schmidt operator norm is denoted by $|\cdot|$. Furthermore, ∇ denotes the canonical gradient differential operator and for a function $h(x,y) : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^n$, we write $\nabla_x h$ or $\nabla_y h$ for the derivatives with respect to x and y. We work with the following spaces:

- For $p \geq 2$, let $L^p(\mathbb{R}^m)$ be the space of $\mathcal{F}_T$-measurable random variables $\xi : \Omega \to \mathbb{R}^m$ normed by $\|\xi\|_{L^p} := \mathbb{E}\big[\,|\xi|^p\,\big]^{1/p}$.
- For $\beta \geq 0$ and $p \geq 1$, $\mathcal{H}^p_\beta(\mathbb{R}^{m\times d})$ denotes the space of all predictable process φ with values in $\mathbb{R}^{m\times d}$ such that the norm $\|\varphi\|_{\mathcal{H}^p_\beta} := \mathbb{E}\Big[\Big(\int_0^T e^{\beta s}|\varphi_s|^2 \mathrm{d}s\Big)^{p/2}\Big]^{1/p} < \infty$.
- For $\beta \geq 0$ and $p \geq 2$, $\mathcal{S}^p_\beta(\mathbb{R}^{m\times d})$ denotes the space of all predictable processes η with values in $\mathbb{R}^{m\times d}$ such that the norm $\|\eta\|_{\mathcal{S}^p_\beta} := \mathbb{E}\Big[\Big(\sup_{0\leq t\leq T} e^{\beta t}|\eta_t|^2\Big)^{p/2}\Big]^{1/p} < \infty$.

We omit referencing the range space if no ambiguity arises. It is fairly easy to see that for any $\beta, \bar\beta \geq 0$ the norms on $\mathcal{H}^p_\beta$, $\mathcal{H}^p_{\bar\beta}$ and $\mathcal{S}^p_\beta$, $\mathcal{S}^p_{\bar\beta}$ are equivalent.

Some notation

We introduce a notational convention which will be used throughout this chapter: for an arbitrarily given integrable function $f : [0,T] \to \mathbb{R}^m$, trivially extended to $[-T, 0)$

via $f(t)\mathbb{1}_{[-T,0)}(t) = 0$, and a given deterministic finite measure α supported on $[-T,0)$ which is not necessarily atomless, we denote for $t \in [0,T]$ and any $p \geq 2$

$$(f \cdot \alpha)(t) := \int_{-T}^{0} f(t+v)\alpha(\mathrm{d}v) \quad \text{and} \quad (f^p \cdot \alpha)(t) := \int_{-T}^{0} |f(t+v)|^p \alpha(\mathrm{d}v).$$

Similarly, for a given process $(\varphi_t)_{t\in[0,T]}$, extended to $[-T,0)$ by imposing $\varphi_t = 0$ on $[-T,0)$, we denote

$$(\varphi \cdot \alpha)(t) := \int_{-T}^{0} \varphi_{t+v}\alpha(\mathrm{d}v), \qquad t \in [0,T], \tag{5.1}$$

and

$$(\varphi^p \cdot \alpha)(t) := \int_{-T}^{0} |\varphi_{t+v}|^p \alpha(\mathrm{d}v), \qquad t \in [0,T],\ p \geq 2. \tag{5.2}$$

We now give a "change of integration order" lemma concerning (5.1) and (5.2) which is used throughout this chapter.

Lemma 5.1.1. *Let φ be a process and α a non-random finite measure supported on $[-T,0)$. Then, we have the following change of integration order: for every $k \geq 1$*

$$\int_t^T (\varphi^k \cdot \alpha)(s)\mathrm{d}s = \int_0^T \alpha\big([r-T,(r-t)\wedge 0)\big)|\varphi_r|^k \mathrm{d}r, \quad \forall t \in [0,T],\ \mathbb{P}-a.s.$$

Moreover, if we have for $p \geq 1$ that $\varphi \in \mathcal{H}_0^p$, then we also have

$$\|(\varphi \cdot \alpha)\|^p_{\mathcal{H}^p_\beta} \leq M_p \|\varphi\|^p_{\mathcal{H}^p_0},$$

where $M_p = (e^{\beta T})^{p/2}\big(\alpha([-T,0))\big)^p$.

Proof. Let $t \in [0,T]$ and $k \in [1,+\infty)$. The first claim follows from

$$\begin{aligned}
\int_t^T (\varphi^k \cdot \alpha)(s)\mathrm{d}s &= \int_t^T \int_{-T}^0 |\varphi_{s+v}|^k \alpha(\mathrm{d}v)\mathrm{d}s = \int_{-T}^0 \int_t^T |\varphi_{s+v}|^k \ \mathrm{d}s\ \alpha(\mathrm{d}v) \\
&= \int_{-T}^0 \int_{(t+v)\vee 0}^{T+v} |\varphi_r|^k \ \mathrm{d}r\ \alpha(\mathrm{d}v) = \int_0^T \int_{(r-T)}^{(r-t)\wedge 0} |\varphi_r|^k \ \alpha(\mathrm{d}v)\ \mathrm{d}r \\
&= \int_0^T \alpha\big([r-T,(r-t)\wedge 0)\big)|\varphi_r|^k \mathrm{d}r.
\end{aligned}$$

The second claim follows by applying Jensen's inequality and changing the integration order as done above, i.e. for any $\beta \geq 0$ and $p \geq 1$ we have

$$\begin{aligned}
\mathbb{E}\bigg[\Big(\int_0^T e^{\beta s}|(\varphi\cdot\alpha)(s)|^2\mathrm{d}s\Big)^{p/2}\bigg] &\leq \big(e^{\beta T}\alpha([-T,0))\big)^{p/2}\ \mathbb{E}\bigg[\Big(\int_0^T (|\varphi|^2\cdot\alpha)(s)\mathrm{d}s\Big)^{p/2}\bigg] \\
&\leq M_p \mathbb{E}\bigg[\Big(\int_0^T |\varphi_s|^2\mathrm{d}s\Big)^{p/2}\bigg] = M_p\|\varphi\|^p_{\mathcal{H}^p_0},
\end{aligned}$$

which concludes the proof. □

5.2 General results on BSDE with time delayed generators

In this section we give a brief overview of BSDEs with time delayed generators and discuss the setting in which they are studied. We then establish convenient a priori estimates for the difference of two solutions to such equations which will play a central role in proving existence and uniqueness of solutions in the more general L^p-spaces.

5.2.1 BSDEs with time delayed generators

Let us start with a recap on BSDEs with time delayed generators. Throughout the chapter, we assume

(H0) $\alpha_{\mathcal{Y}}, \alpha_{\mathcal{Z}}$ are two non-random, finitely valued measures supported on $[-T, 0)$.

We also define

$$\alpha := \alpha_{\mathcal{Y}}([-T,0)) \vee \alpha_{\mathcal{Z}}([-T,0)). \tag{5.3}$$

Given $p \geq 2$, we assume that the following holds:

(H1) ξ is an $\mathcal{F}_T$-measurable random variable which belongs to $L^p(\mathbb{R}^m)$;

(H2) the generator $f : \Omega \times [0,T] \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^m$ is measurable, $\mathbb{F}$-adapted and satisfies the following Lipschitz type condition: there exists a constant $K > 0$ such that

$$\left|f(t,y,z) - f(t,y',z')\right|^2 \leq K\left(|y-y'|^2 + |z-z'|^2\right)$$

holds for $\mathrm{d}\mathbb{P}\otimes\mathrm{d}t$-almost all $(\omega,t) \in \Omega\times[0,T]$ and for every $(y,z), (y'z') \in \mathbb{R}^m \times \mathbb{R}^{m\times d}$;

(H3) $\mathbb{E}\Big[\big(\int_0^T |f(s,0,0)|^2 \mathrm{d}s\big)^{p/2}\Big] < \infty$;

(H4) $f(t,\cdot,\cdot) = 0$ if $t < 0$.

Following the notation from equation (5.1), we write

$$(Y\cdot\alpha_{\mathcal{Y}})(t) = \int_{-T}^0 Y_{t+v}\alpha_{\mathcal{Y}}(\mathrm{d}v) \quad \text{and} \quad (Z\cdot\alpha_{\mathcal{Z}})(t) = \int_{-T}^0 Z_{t+v}\alpha_{\mathcal{Z}}(\mathrm{d}v), \quad 0 \leq t \leq T,$$

for some processes $(Y_t)_{t\in[0,T]}$ and $(Z_t)_{t\in[0,T]}$ satisfying appropriate integrability conditions. Assumption (H2) and Jensen's inequality then imply

$$\begin{aligned}\text{(H2')}\quad &\left|f\big(t,(Y\cdot\alpha_{\mathcal{Y}})(t),(Z\cdot\alpha_{\mathcal{Z}})(t)\big) - f\big(t,(Y'\cdot\alpha_{\mathcal{Y}})(t),(Z'\cdot\alpha_{\mathcal{Z}})(t)\big)\right|^2\\ &\leq K\Big\{\left|\big((Y-Y')\cdot\alpha_{\mathcal{Y}}\big)(t)\right|^2 + \left|\big((Z-Z')\cdot\alpha_{\mathcal{Z}}\big)(t)\right|^2\Big\}\\ &\leq L\Big\{\big((Y-Y')^2\cdot\alpha_{\mathcal{Y}}\big)(t) + \big((Z-Z')^2\cdot\alpha_{\mathcal{Z}}\big)(t)\Big\},\end{aligned}$$

where $L \coloneqq K\alpha$ with the real number α given by (5.3). The focus of our study are BSDEs with time delayed generators which are of the type

$$Y_t = \xi + \int_t^T f\big(s, \Gamma(s)\big)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \quad 0 \le t \le T, \tag{5.4}$$

where Γ abbreviates for $t \in [0,T]$ the following expression

$$\Gamma(t) \coloneqq \Big(\int_{-T}^0 Y_{t+v}\alpha_{\mathcal{Y}}(\mathrm{d}v), \int_{-T}^0 Z_{t+v}\alpha_{\mathcal{Z}}(\mathrm{d}v) \Big) = \Big((Y \cdot \alpha_{\mathcal{Y}})(t), (Z \cdot \alpha_{\mathcal{Z}})(t) \Big). \tag{5.5}$$

Definition 5.2.1 (Solution of a Delay BSDE)**.** *We say that (Y,Z) is a solution to the delay BSDE* (5.4) *if (Y,Z) belongs to the space $\mathcal{S}_0^p \times \mathcal{H}_0^p$ and satisfies* (5.4)*.*

Using a fixed point argument, Delong and Imkeller [42] have shown that a BSDE of the type (5.4)-(5.5) admits a unique solution if the parameters of the equation (5.4) are sufficiently small, i.e. if the Lipschitz constant $K > 0$ or the terminal time $T > 0$ satisfy a smallness condition. The following L^2-existence and uniqueness result is a straightforward modification of Theorem 2.1 from Delong and Imkeller [42].

Theorem 5.2.1. *Let $p = 2$ and assume that (H0)-(H4) are satisfied. For α defined as in* (5.3)*, assume that the non-negative constants T, $L = K\alpha$, β are such that*

$$(8T + \frac{1}{\beta})L \int_{-T}^0 e^{-\beta u}\rho(\mathrm{d}u) \max\{1, T\} < 1, \quad \textit{for } \rho \in \{\alpha_{\mathcal{Y}}, \alpha_{\mathcal{Z}}\}.$$

Then, the delay BSDE (5.4)-(5.5) *has a unique solution $(Y,Z) \in \mathcal{S}_\beta^2(\mathbb{R}^m) \times \mathcal{H}_\beta^2(\mathbb{R}^{m\times d})$.*

Remark 5.2.1. *In Delong and Imkeller [42], this result is proved for the case of one-dimensional components, i.e. $d = m = 1$. It is clear that by the nature of the fixed point argument, the proof is insensitive with respect to the dimension of Y or Z.*

Given that a compatibility condition is necessary in order to establish existence and uniqueness of solutions and moreover that we will be giving an extended version of it, all the proofs in this chapter are given with extra detail in order to better control the constants involved in abounding estimates.

5.2.2 Moment and a priori estimates

In Lemma 2.1 from Delong and Imkeller [42] the authors provide a priori estimates for the time delayed BSDE (5.4) which estimates the norms of the difference between the solution of two BSDEs in terms of the terminal conditions and the difference of the generators applied to the solution processes. More specifically, for $i \in \{1,2\}$ let (Y^i, Z^i) be solutions of BSDEs, driven by the dynamics from (5.4), with terminal conditions ξ^i

and drivers f^i satisfying (H1)-(H4). Then we have

$$\|Y^1 - Y^2\|^2_{\mathcal{H}^2_\beta} + \|Z^1 - Z^2\|^2_{\mathcal{H}^2_\beta} \le C_2\Big\{\mathbb{E}\big[\, e^{\beta T}|Y^1_T - Y^2_T|^2\,\big] + \mathbb{E}\Big[\int_0^T e^{\beta s}|f^1(s,(Y^1\cdot\alpha)(s),(Z^1\cdot\alpha)(s)) - f^2(s,(Y^2\cdot\alpha)(s),(Z^2\cdot\alpha)(s))|^2\mathrm{d}s\,\Big]\Big\}, \tag{5.6}$$

where the authors assume that α is some deterministic measure on $[-T,0)$ with mass one. Thus Lemma 2.1 from Delong and Imkeller [42] establishes the a priori estimate (5.6) whose right-hand side depends on the solution of *both* delay BSDEs. In the context of Delong and Imkeller [42] such a result suffices to establish existence and uniqueness of solutions in $\mathcal{S}^2_\beta \times \mathcal{H}^2_\beta$ but the situation becomes more intricate when the same issues are considered on $\mathcal{S}^p_\beta \times \mathcal{H}^p_\beta$ for $p > 2$. More precisely, we are not able to obtain an estimate similar to (5.6) when $p > 2$. In the study of differentiability of the solution (for both $p = 2$ and $p > 2$) in Section 5.3, it turns out that we require a priori estimates where the right-hand side of the estimate depends only on the problem's data: the difference between the terminal conditions and a quantity of the form $\delta_2 f_s := f^1(s,(Y^2\cdot\alpha)(s),(Z^2\cdot\alpha)(s)) - f^2(s,(Y^2\cdot\alpha)(s),(Z^2\cdot\alpha)(s))$. For a clear view of the required estimates, compare for instance (5.6) with (5.9).

Moment estimates - part I

As a starting observation, we see that if (5.4) admits a solution (Y,Z) in $\mathcal{H}^p_\beta(\mathbb{R}^m)\times\mathcal{H}^p_\beta(\mathbb{R}^{m\times d})$, then $Y \in \mathcal{S}^p_\beta(\mathbb{R}^m)$.

Lemma 5.2.1. *Let $\beta \ge 0$, $p \ge 2$ and assume that (H0)-(H4) hold. If the delay BSDE* (5.4) *admits a solution $(Y,Z) \in \mathcal{H}^p_\beta(\mathbb{R}^m)\times\mathcal{H}^p_\beta(\mathbb{R}^{m\times d})$, then we have in particular $Y \in \mathcal{S}^p_\beta(\mathbb{R}^m)$.*

Proof. Throughout let $t \in [0,T]$ and $p \ge 2$. Since all β-norms are equivalent, it suffices to show the result for $\beta = 0$. We drop the β-subscripts in the following. The pair (Y,Z) satisfies

$$Y_t = \xi + \int_t^T f\big(s,(Y\cdot\alpha_{\mathcal{Y}})(s),(Z\cdot\alpha_{\mathcal{Z}})(s)\big)\mathrm{d}s - \int_t^T Z_s\mathrm{d}W_s,$$

which yields

$$\sup_{0\le t\le T}|Y_t| \le |\xi| + \int_0^T \big|f\big(s,(Y\cdot\alpha_{\mathcal{Y}})(s),(Z\cdot\alpha_{\mathcal{Z}})(s)\big)\big|\mathrm{d}s + \sup_{0\le t\le T}\Big|\int_t^T Z_s\mathrm{d}W_s\Big|.$$

Combining $Z \in \mathcal{H}^p$ with the inequalities by Young, Doob and Burkholder-Davis-Gundy

(BDG), we obtain

$$\mathbb{E}\Big[\Big(\sup_{0\le t\le T}\Big|\int_t^T Z_s \mathrm{d}W_s\Big|^2\Big)^{p/2}\Big] \le 2^{p/2}\,\mathbb{E}\Big[\Big(\Big|\int_0^T Z_s \mathrm{d}W_s\Big|^2 + \sup_{0\le t\le T}\Big|\int_0^t Z_s \mathrm{d}W_s\Big|^2\Big)^{p/2}\Big]$$
$$\le 2^p\,\mathbb{E}\Big[\sup_{0\le t\le T}\Big|\int_0^t Z_s \mathrm{d}W_s\Big|^p\Big] \le 2^p C_p \|Z\|^p_{\mathcal{H}^p_0} < \infty.$$

Next observe that by the Lipschitz property of the generator f (note that (H2) implies (H2')), it follows that

$$\Big(\int_0^T \big|f\big(s,(Y\cdot\alpha_{\mathcal{Y}})(s),(Z\cdot\alpha_{\mathcal{Z}})(s)\big)\big|^2\mathrm{d}s\Big)^{p/2}$$
$$\le 2^{p/2}\Big(\int_0^T \big|f(s,0,0)\big|^2\mathrm{d}s + \int_0^T \Big|f\big(s,(Y\cdot\alpha_{\mathcal{Y}})(s),(Z\cdot\alpha_{\mathcal{Z}})(s)\big) - f(s,0,0)\Big|^2\mathrm{d}s\Big)^{p/2}$$
$$\le 2^{p/2}2^{p/2-1}\Big\{\Big(\int_0^T \big|f(s,0,0)\big|^2\mathrm{d}s\Big)^{p/2} + \Big(L\int_0^T\Big((|Y|^2\cdot\alpha_{\mathcal{Y}})(s) + (|Z|^2\cdot\alpha_{\mathcal{Z}})(s)\Big)\mathrm{d}s\Big)^{p/2}\Big\}.$$

The second term in the bracket can be further estimated by

$$\Big(L\int_0^T\Big((|Y|^2\cdot\alpha_{\mathcal{Y}})(s) + (|Z|^2\cdot\alpha_{\mathcal{Z}})(s)\Big)\mathrm{d}s\Big)^{p/2}$$
$$\le 2^{p/2-1}L^{p/2}\Big\{\Big(\int_0^T (|Y|^2\cdot\alpha_{\mathcal{Y}})(s)\mathrm{d}s\Big)^{p/2} + \Big(\int_0^T (|Z|^2\cdot\alpha_{\mathcal{Z}})(s)\mathrm{d}s\Big)^{p/2}\Big\}$$
$$\le 2^{p/2-1}L^{p/2}\alpha^{p/2}\Big\{\Big(\int_0^T |Y_s|^2\mathrm{d}s\Big)^{p/2} + \Big(\int_0^T |Z_s|^2\mathrm{d}s\Big)^{p/2}\Big\},$$

where the last line follows from Lemma 5.1.1. This estimate together with (H3) yields

$$\mathbb{E}\Big[\Big(\int_0^T \big|f\big(s,(Y\cdot\alpha_{\mathcal{Y}})(s),(Z\cdot\alpha_{\mathcal{Z}})(s)\big)\big|^2\mathrm{d}s\Big)^{p/2}\Big] < \infty.$$

Using hypothesis (H1), i.e. $\xi\in L^p$, we conclude that we have $Y\in\mathcal{S}^p$. □

A priori estimates

Let us define the weighted variant $\tilde{\alpha}$ of α as the maximum of the weighted measures $\alpha_{\mathcal{Y}}$ and $\alpha_{\mathcal{Z}}$ on $[-T,0)$ by

$$\tilde{\alpha} := \int_{-T}^0 e^{-\beta s}\alpha_{\mathcal{Y}}(\mathrm{d}s) \vee \int_{-T}^0 e^{-\beta s}\alpha_{\mathcal{Z}}(\mathrm{d}s), \quad \beta\ge 0. \tag{5.7}$$

Remark 5.2.2. *We emphasize that $\tilde{\alpha}$ depends on β. To avoid notational overload, we write $\tilde{\alpha}$ and skip the explicit dependence.*

The next results establish *canonical* a priori estimates for the solutions of two time-delayed BSDEs as given by (5.4). *Canonical* is meant in the sense that the right-hand side of the estimate only depends on the problem's data. We distinguish the cases $p = 2$ and $p > 2$ whose proofs deviate in their degree of technicality. We start with the case $p = 2$.

Proposition 5.2.1 (A priori estimates for $p = 2$)**.** *Let $p = 2$. Consider $i \in \{1, 2\}$ and let $(Y^i, Z^i) \in \mathcal{S}_0^2 \times \mathcal{H}_0^2$ be the solution of the delay BSDE* (5.4) *with terminal condition ξ^i and generator f^i satisfying (H0)-(H4). Denote by $K > 0$ the Lipschitz constant of f^1 as given in (H2') and set $\delta Y = Y^1 - Y^2$, $\delta Z = Z^1 - Z^2$. If either T or K or α are small enough, then there exist two constants $\beta, \gamma > 0$ satisfying*

$$D_1 := \beta - \gamma - \frac{\tilde{\alpha} L}{\gamma} > 0 \quad \textit{and} \quad D_2 := 1 - \frac{\tilde{\alpha} L}{\gamma} > 0 \qquad (\textit{with } L = K\alpha \textit{ and } \alpha \textit{ as in } (5.3)), \tag{5.8}$$

and a constant $C_2 = C_2(\beta, \gamma, \tilde{\alpha}, L, T) > 0$ depending on $\beta, \gamma, \tilde{\alpha}, L, T$ such that for $i \in \{1, 2\}$, we have $(Y^i, Z^i) \in \mathcal{S}_\beta^2 \times \mathcal{H}_\beta^2$

$$\|\delta Y\|^2_{\mathcal{S}^2_\beta} + \|\delta Y\|^2_{\mathcal{H}^2_\beta} + \|\delta Z\|^2_{\mathcal{H}^2_\beta} \le C_2 \Big\{ \mathbb{E}\Big[e^{\beta T} |\delta Y_T|^2 \Big] + \mathbb{E}\Big[\int_0^T e^{\beta s} |\delta_2 f_s|^2 \mathrm{d}s \Big] \Big\}, \tag{5.9}$$

where $\delta_2 f_t := f^1\big(t, (Y^2 \cdot \alpha_{\mathcal{Y}})(t), (Z^2 \cdot \alpha_{\mathcal{Y}})(t)\big) - f^2\big(t, (Y^2 \cdot \alpha_{\mathcal{Y}})(t), (Z^2 \cdot \alpha_{\mathcal{Y}})(t)\big)$ for $t \in [0, T]$.

Proof. Let γ, K, T, α be such that the relations in (5.8) are satisfied (i.e. $D_1 > 0$ and $D_2 > 0$). Throughout the proof, let $t \in [0, T]$, $i \in \{1, 2\}$ and define Γ^i as in (5.5) for the pair (Y^i, Z^i). An application of Itô's formula to $e^{\beta t} |\delta Y_t|^2$ for $\beta > 0$ yields

$$\begin{aligned}
e^{\beta t} |\delta Y_t|^2 &+ \int_t^T \beta e^{\beta s} |\delta Y_s|^2 \mathrm{d}s + \int_t^T e^{\beta s} |\delta Z_s|^2 \mathrm{d}s \\
&= e^{\beta T} |\delta Y_T|^2 + \int_t^T 2 e^{\beta s} \big\langle \delta Y_s, f^1(s, \Gamma^1(s)) - f^2(s, \Gamma^2(s)) \big\rangle \mathrm{d}s - \int_t^T 2 e^{\beta s} \langle \delta Y_s, \delta Z_s \mathrm{d}W_s \rangle \\
&\le e^{\beta T} |\delta Y_T|^2 + \int_t^T \gamma e^{\beta s} |\delta Y_s|^2 \mathrm{d}s + \int_t^T \frac{e^{\beta s}}{\gamma} \Big(\big| f^1(s, \Gamma^1(s)) - f^1(s, \Gamma^2(s)) \big|^2 \Big) \mathrm{d}s \\
&\qquad + 2 \int_t^T e^{\beta s} \big\langle \delta Y_s, \delta_2 f_s \big\rangle \mathrm{d}s - \int_t^T 2 e^{\beta s} \langle \delta Y_s, \delta Z_s \mathrm{d}W_s \rangle,
\end{aligned}$$

where the last inequality results from Young's inequality for γ. Reorganizing and taking condition (H2') for the generator f^1 into account, we get

$$\begin{aligned}
e^{\beta t} |\delta Y_t|^2 &+ \int_t^T (\beta - \gamma) e^{\beta s} |\delta Y_s|^2 \mathrm{d}s + \int_t^T e^{\beta s} |\delta Z_s|^2 \mathrm{d}s \\
&\le e^{\beta T} |\delta Y_T|^2 + \int_t^T \frac{e^{\beta s}}{\gamma} L \Big[(|\delta Y|^2 \cdot \alpha_{\mathcal{Y}})(s) + (|\delta Z|^2 \cdot \alpha_{\mathcal{Z}})(s) \Big] \mathrm{d}s \\
&\qquad + 2 \int_t^T e^{\beta s} \big\langle \delta Y_s, \delta_2 f_s \big\rangle \mathrm{d}s - \int_t^T 2 e^{\beta s} \langle \delta Y_s, \delta Z_s \rangle \mathrm{d}W_s.
\end{aligned}$$

By a change of integration order argument similar as in the proof of Lemma 5.1.1, we obtain for $j \in \{\mathcal{Y}, \mathcal{Z}\}$ and $\phi^{\mathcal{Y}} = \delta Y$, $\phi^{\mathcal{Z}} = \delta Z$

$$\begin{aligned}
&\int_t^T e^{\beta s}(|\phi^j|^2 \cdot \alpha_j)(s)\mathrm{d}s \\
&\quad = \int_t^T \int_{-T}^0 e^{\beta(s+v)} e^{-\beta v} \mathbb{1}_{\{s+v\geq 0\}} |\phi^j_{s+v}|^2 \alpha_j(\mathrm{d}v)\mathrm{d}s \\
&\quad = \int_{-T}^0 \int_{(t+v)\vee 0}^{T+v} e^{\beta r} e^{-\beta v} \mathbb{1}_{\{r\geq 0\}} |\phi^j_r|^2 \mathrm{d}r \ \alpha_j(\mathrm{d}v) = \int_0^T \int_{r-T}^{(r-t)\wedge 0} e^{\beta r} e^{-\beta v} |\phi^j_r|^2 \alpha_j(\mathrm{d}v) \ \mathrm{d}r \\
&\quad \leq \int_0^T e^{\beta r} |\phi^j_r|^2 \Big(\int_{-T}^0 e^{-\beta v} \alpha_j(\mathrm{d}v)\Big)\mathrm{d}r \leq \int_0^T \tilde{\alpha} e^{\beta r} |\phi^j_r|^2 \mathrm{d}r, \qquad (5.10)
\end{aligned}$$

with $\tilde{\alpha}$ given by (5.7). Continuing the inequality from above we get

$$\begin{aligned}
e^{\beta t}|\delta Y_t|^2 + \int_t^T (\beta - \gamma) e^{\beta s} |\delta Y_s|^2 \mathrm{d}s + \int_t^T e^{\beta s} |\delta Z_s|^2 \mathrm{d}s \leq e^{\beta T} |\delta Y_T|^2 + 2 \int_t^T e^{\beta s} \big\langle \delta Y_s, \delta_2 f_s \big\rangle \mathrm{d}s \\
+ \int_0^T \frac{\tilde{\alpha} L}{\gamma} e^{\beta s} \Big(|\delta Y_s|^2 + |\delta Z_s|^2 \Big) \mathrm{d}s - \int_t^T 2 e^{\beta s} \langle \delta Y_s, \delta Z_s \mathrm{d}W_s \rangle. \qquad (5.11)
\end{aligned}$$

Taking expectations at $t = 0$ yields

$$\begin{aligned}
&\big(\beta - \gamma - \frac{\tilde{\alpha} L}{\gamma}\big) \mathbb{E}\Big[\int_0^T e^{\beta s} |\delta Y_s|^2 \mathrm{d}s \Big] + \big(1 - \frac{\tilde{\alpha} L}{\gamma}\big) \mathbb{E}\Big[\int_0^T e^{\beta s} |\delta Z_s|^2 \mathrm{d}s \Big] \\
&\quad \leq \mathbb{E}\Big[e^{\beta T} |\delta Y_T|^2 \Big] + 2 \mathbb{E}\Big[\int_0^T e^{\beta s} \big\langle \delta Y_s, \delta_2 f_s \big\rangle \mathrm{d}s \Big] \\
&\quad \leq \mathbb{E}\Big[e^{\beta T} |\delta Y_T|^2 \Big] + 2 \mathbb{E}\Big[\sup_{0\leq t\leq T} e^{\frac{\beta}{2} t} |\delta Y_t| \int_0^T e^{\frac{\beta}{2} s} |\delta_2 f_s| \mathrm{d}s \Big] \\
&\quad \leq \mathbb{E}\Big[e^{\beta T} |\delta Y_T|^2 \Big] + \gamma' \mathbb{E}\Big[\sup_{0\leq t\leq T} e^{\beta t} |\delta Y_t|^2 \Big] + \frac{1}{\gamma'} \mathbb{E}\Big[\big(\int_0^T e^{\frac{\beta}{2} s} |\delta_2 f_s| \mathrm{d}s \big)^2 \Big],
\end{aligned}$$

where we have used Young's inequality with some $\gamma' > 0$ to be specified later. By the last expression and by $D_1, D_2 > 0$ (see (5.8)) we get

$$\|\delta Y\|^2_{\mathcal{H}^2_\beta} + \|\delta Z\|^2_{\mathcal{H}^2_\beta} \leq C \Big\{ \mathbb{E}\Big[e^{\beta T} |\delta Y_T|^2 \Big] + \gamma' \|\delta Y\|^2_{\mathcal{S}^2_\beta} + \frac{1}{\gamma'} \mathbb{E}\Big[\big(\int_0^T e^{\frac{\beta}{2} s} |\delta_2 f_s| \mathrm{d}s \big)^2 \Big\}, \qquad (5.12)$$

where $C > 0$ is a constant depending $\beta, \gamma, \tilde{\alpha}, L$ and T. In order to obtain the $\mathcal{S}^2_\beta$-estimate for δY we observe that we have

$$\delta Y_t \leq \delta Y_T + \int_t^T \big| f^1\big(s, \Gamma^1(s)\big) - f^1\big(s, \Gamma^2(s)\big) \big| \mathrm{d}s + \int_t^T \big| \delta_2 f_s \big| \mathrm{d}s - \int_t^T \delta Z_s \mathrm{d}W_s.$$

Multiplying by the monotone increasing function $e^{\frac{\beta}{2} t}$ and taking the conditional expectation with respect to $\mathcal{F}_t$ we get

$$\begin{aligned}
e^{\frac{\beta}{2}t}\delta Y_t &\leq \mathbb{E}\Big[e^{\frac{\beta}{2}t}|\delta Y_T| + e^{\frac{\beta}{2}t}\int_t^T \big|f^1\big(s,\Gamma^1(s)\big) - f^1\big(s,\Gamma^2(s)\big)\big|\mathrm{d}s + e^{\frac{\beta}{2}t}\int_t^T |\delta_2 f_s|\mathrm{d}s\big|\mathcal{F}_t\Big]\\
&\leq \mathbb{E}\Big[e^{\frac{\beta}{2}T}|\delta Y_T| + \int_t^T e^{\frac{\beta}{2}s}\big|f^1\big(s,\Gamma^1(s)\big) - f^1\big(s,\Gamma^2(s)\big)\big|\mathrm{d}s\\
&\qquad + \int_0^t e^{\frac{\beta}{2}s}\big|f^1\big(s,\Gamma^1(s)\big) - f^1\big(s,\Gamma^2(s)\big)\big|\mathrm{d}s + \int_t^T e^{\frac{\beta}{2}s}|\delta_2 f_s|\mathrm{d}s\\
&\qquad + \int_0^t e^{\frac{\beta}{2}s}|\delta_2 f_s|\mathrm{d}s\big|\mathcal{F}_t\Big]\\
&= \mathbb{E}\Big[e^{\frac{\beta}{2}T}|\delta Y_T| + \int_0^T e^{\frac{\beta}{2}s}\big|f^1\big(s,\Gamma^1(s)\big) - f^1\big(s,\Gamma^2(s)\big)\big|\mathrm{d}s + \int_0^T e^{\frac{\beta}{2}s}|\delta_2 f_s|\mathrm{d}s\big|\mathcal{F}_t\Big].
\end{aligned}$$

Using Doob's inequality, we obtain

$$\begin{aligned}
&\|\delta Y\|^2_{\mathcal{S}^2_\beta}\\
&\leq 4\ \mathbb{E}\Big[\Big(\mathbb{E}\Big[e^{\frac{\beta}{2}T}|\delta Y_T| + \int_0^T e^{\frac{\beta}{2}s}\big|f^1\big(s,\Gamma^1(s)\big) - f^1\big(s,\Gamma^2(s)\big)\big|\mathrm{d}s + \int_0^T e^{\frac{\beta}{2}s}|\delta_2 f_s|\mathrm{d}s \mid \mathcal{F}_T\Big]\Big)^2\Big]\\
&\leq 12\ \mathbb{E}\Big[e^{\beta T}|\delta Y_T|^2 + T\int_0^T e^{\beta s}\big|f^1\big(s,\Gamma^1(s)\big) - f^1\big(s,\Gamma^2(s)\big)\big|^2\mathrm{d}s + \Big(\int_0^T e^{\frac{\beta}{2}s}|\delta_2 f_s|\mathrm{d}s\Big)^2\Big],
\end{aligned}$$

where the last line follows by Jensen's inequality. Since f^1 satisfies (H2'), an application of Lemma 5.1.1 yields

$$\|\delta Y\|^2_{\mathcal{S}^2_\beta} \leq 12\Big\{\mathbb{E}\Big[e^{\beta T}|\delta Y_T|^2\Big] + \tilde{\alpha}TL\Big(\|\delta Y\|^2_{\mathcal{H}^2_\beta} + \|\delta Z\|^2_{\mathcal{H}^2_\beta}\Big) + \mathbb{E}\Big[\Big(\int_0^T e^{\frac{\beta}{2}s}|\delta_2 f_s|\mathrm{d}s\Big)^2\Big]\Big\}.$$

Hence, plugging into (5.12) we find

$$\begin{aligned}
&\big(1 - 12C\gamma'\tilde{\alpha}TL\big)\mathbb{E}\Big[\sup_{0\leq t\leq T} e^{\beta t}|\delta Y_t|^2\Big]\\
&\qquad \leq 12\Big\{\big(1 + C\tilde{\alpha}TL\big)\mathbb{E}\Big[e^{\beta T}|\delta Y_T|^2\Big] + \big(1 + C\gamma'^{-1}\tilde{\alpha}TL\big)\mathbb{E}\Big[\Big(\int_0^T e^{\frac{\beta}{2}s}|\delta_2 f_s|\mathrm{d}s\Big)^2\Big]\Big\}.
\end{aligned}$$

Choosing γ' small enough such that $(1 - 12C\gamma'\tilde{\alpha}TL) > 0$ is satisfied we conclude that estimate (5.9) holds for a constant $C_2 = C_2(\beta,\gamma,\tilde{\alpha},L,T)$. □

Remark 5.2.3. *Note that in the previous result we have three degrees of freedom: the Lipschitz constant of the driver K, the time horizon T and the duration of the time delay given by α.*

The proof for the case $p > 2$ is more involved and uses techniques from the proof of Proposition 5.2.1. The main reason for the increase of technicality lies in (5.11). Usually the dynamics of Y is described by integrals over the interval $[t,T]$. However, for delay BSDEs we see from (5.11) that the dynamics of Y also depends on a integral over the whole interval $[0,T]$. We also mention that the techniques from Delong and Imkeller

[42] cannot be extended in L^p (for $p > 2$). For more details, we refer to estimate (2.3) in proof of Lemma 2.1 from Delong and Imkeller [42].

The next proposition gives a result which will be central in establishing existence and uniqueness of L^p solutions to delay BSDEs as well as in proving the differentiability results in Section 5.3.

Proposition 5.2.2 (A priori estimates for $p > 2$)**.** *Let $p > 2$. Consider $i \in \{1,2\}$ and denote by $(Y^i, Z^i) \in \mathcal{S}_0^p \times \mathcal{H}_0^p$ a solution of the delay BSDE (5.4) with terminal condition ξ^i and generator f^i satisfying (H0)-(H4). Denote by $K > 0$ the Lipschitz constant of f^1 in (H2') and set $\delta Y = Y^1 - Y^2$, $\delta Z = Z^1 - Z^2$. If either T or K or α are small enough (for $L = K\alpha$, α as in (5.3) and $\tilde{\alpha}$ as in (5.7)) then there exists $\beta, \gamma > 0$ satisfying (5.8) (i.e. $D_1, D_2 > 0$) and*

$$D_3 := 1 - 2^{4p-4} d_{p/2}^2 \Big(\frac{p}{p-2}\Big)^{p/2} \Big(\frac{\tilde{\alpha}L}{\gamma - \tilde{\alpha}L}\Big)^{p/2} D_2^{-p/2} - \Big(\frac{\tilde{\alpha}L}{\gamma} T\Big)^{p/2} \Big(\frac{p}{p-2}\Big)^{p/2} 2^{p-2} > 0 \qquad (5.13)$$

where $m \in \mathbb{N}$ denotes the dimension of the δY process and the constant $d_{p/2}$ is given by

$$d_{p/2} := m^{p/2+1} \Big(\frac{p}{p-1}\Big)^{p^2/2} \Big(\frac{p(p-1)}{2}\Big)^{p/2}. \qquad (5.14)$$

In addition, we have $(Y^i, Z^i) \in \mathcal{S}_\beta^p \times \mathcal{H}_\beta^p$ for $i \in \{1,2\}$ and there exists a constant $C_p = C_p(\beta, \gamma, \tilde{\alpha}, L, T, m) > 0$ explicitly given in (5.26) such that

$$\|\delta Y\|_{\mathcal{S}_\beta^p}^p + \|\delta Y\|_{\mathcal{H}_\beta^p}^p + \|\delta Z\|_{\mathcal{H}_\beta^p}^p \le C_p \Big\{ \mathbb{E}\Big[\big(e^{\beta T} |\delta Y_T|^2\big)^{p/2} \Big] + \mathbb{E}\Big[\Big(\int_0^T e^{\frac{\beta}{2} s} |\delta_2 f_s| \mathrm{d}s \Big)^p \Big] \Big\}, \qquad (5.15)$$

with $\delta_2 f_t = f^1\big(t, Y^2(t), Z^2(t)\big) - f^2\big(t, Y^2(t), Z^2(t)\big)$, for $t \in [0,T]$.

Remark 5.2.4. *A closer analysis on the constants D_1, D_2 and D_3 shows*

$$\lim_{K\alpha \to 0} (D_1, D_2, D_3) > (0,0,0).$$

This means that with either a small T or a small K or a small α the conditions of Proposition 5.2.2 can be verified.

Proof of Proposition 5.2.2. Throughout the proof let $t \in [0,T]$, $i \in \{1,2\}$ and from (5.8) define $D_1 := \beta - \gamma - \frac{\tilde{\alpha}L}{\gamma}$ and $D_2 := 1 - \frac{\tilde{\alpha}L}{\gamma}$. We emphasize that $\tilde{\alpha}$ as defined in (5.7) depends on β. Recall (5.11) from the proof of Proposition 5.2.1:

$$\begin{aligned} e^{\beta t} |\delta Y_t|^2 + \int_t^T (\beta - \gamma) e^{\beta s} |\delta Y_s|^2 \mathrm{d}s + \int_t^T e^{\beta s} |\delta Z_s|^2 \mathrm{d}s \le e^{\beta T} |\delta Y_T|^2 + 2 \int_t^T e^{\beta s} \big\langle \delta Y_s, \delta_2 f_s \big\rangle \mathrm{d}s \\ + \int_0^T \frac{\tilde{\alpha}L}{\gamma} e^{\beta s} \Big(|\delta Y_s|^2 + |\delta Z_s|^2 \Big) \mathrm{d}s - \int_t^T 2 e^{\beta s} \langle \delta Y_s, \delta Z_s \mathrm{d}W_s \rangle. \qquad (5.16) \end{aligned}$$

By assumption, the constants $\beta, \gamma, T, K, \alpha$ are such that (5.8) holds and hence we have that $D_1 > 0$ and $D_2 > 0$. We now proceed in several steps.

Step 1: We claim that

$$\mathbb{E}\Big[\Big(\int_0^T e^{\beta s}|\delta Z_s|^2\mathrm{d}s\Big)^{p/2}\Big] \le D_2^{-p/2}\Big\{2^{p/2}\mathbb{E}\Big[\big(e^{\beta T}|\delta Y_T|^2\big)^{p/2}\Big] + 2^{3p-2}d_{p/2}^2 D_2^{-p/2}\|\delta Y\|^p_{\mathcal{S}^p_\beta} + 2^{3p/2-1}\mathbb{E}\Big[\Big|\int_0^T e^{\beta s}\big\langle\delta Y_s,\delta_2 f_s\big\rangle\mathrm{d}s\Big|^{p/2}\Big]\Big\}, \tag{5.17}$$

where $d_{p/2} > 0$ is a given constant appearing in the BDG inequality which only depends on $p > 2$ and the dimension. Estimate (5.17) can be deduced as follows: putting $t = 0$ in (5.16) and observing that by (5.8) the constants D_1 and D_2 are positive we get

$$\begin{aligned}\Big(1-\frac{\tilde{\alpha}L}{\gamma}\Big)\int_0^T e^{\beta s}|\delta Z_s|^2\mathrm{d}s &\le \Big(\beta-\gamma-\frac{\tilde{\alpha}L}{\gamma}\Big)\int_0^T e^{\beta s}|\delta Y_s|^2\mathrm{d}s + \Big(1-\frac{\tilde{\alpha}L}{\gamma}\Big)\int_0^T e^{\beta s}|\delta Z_s|^2\mathrm{d}s \\ &\le e^{\beta T}|\delta Y_T|^2 + 2\int_0^T e^{\beta s}\big\langle\delta Y_s,\delta_2 f_s\big\rangle\mathrm{d}s - 2\int_0^T e^{\beta s}\langle\delta Y_s,\delta Z_s\mathrm{d}W_s\rangle.\end{aligned}$$

Now raising both sides to the power $p/2 > 1$, making use of the fact that for $a, b, c \in \mathbb{R}$

$$\begin{aligned}\big|a+2b-2c\big|^{p/2} &\le 2^{p/2-1}\Big(|a|^{p/2} + |2b-2c|^{p/2}\Big) \le 2^{p/2-1}\Big(|a|^{p/2} + 2^{p/2-1}\big(|2b|^{p/2} + |2c|^{p/2}\big)\Big) \\ &= 2^{p/2-1}|a|^{p/2} + 2^{3p/2-2}|b|^{p/2} + 2^{3p/2-2}|c|^{p/2}\end{aligned}$$

and taking expectations, we get

$$\begin{aligned}\Big(1-\frac{\tilde{\alpha}L}{\gamma}\Big)^{p/2}\,\mathbb{E}\Big[\Big(\int_0^T e^{\beta s}|\delta Z_s|^2\mathrm{d}s\Big)^{p/2}\Big] &\le 2^{p/2-1}\mathbb{E}\Big[\big(e^{\beta T}|\delta Y_T|^2\big)^{p/2}\Big] \\ &+ 2^{3p/2-2}\mathbb{E}\Big[\Big|\int_0^T e^{\beta s}\big\langle\delta Y_s,\delta_2 f_s\big\rangle\mathrm{d}s\Big|^{p/2}\Big] + 2^{3p/2-2}\mathbb{E}\Big[\Big|\int_0^T e^{\beta s}\langle\delta Y_s,\delta Z_s\mathrm{d}W_s\rangle\Big|^{p/2}\Big].\end{aligned} \tag{5.18}$$

Denoting

$$\mathrm{d}N_t^j := \sum_{k=1}^d \delta Z_t^{k,j}\mathrm{d}W_t^k,$$

we apply the BDG inequality with the constant

$$C^* := \Big(\frac{p}{p-1}\Big)^{p^2/2}\Big(\frac{p(p-1)}{2}\Big)^{p/2} > 0,$$

(see Theorem 3.9.1 from Khoshnevisan [75] and solution to Problem 3.29, p. 231, in

Karatzas and Shreve [71]) and Young's inequality with some constant $\gamma_2 > 0$ and obtain

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\int_0^T e^{\beta s}\langle \delta Y_s, \delta Z_s \mathrm{d}W_s\rangle\Big|^{p/2}\Big] \le \mathbb{E}\Big[\Big(\sum_{j=1}^m \Big|\int_0^T e^{\beta s}\delta Y_s^j\ \mathrm{d}N_s^j\Big)^{p/2}\Big] \\
&\le m^{p/2}\ \sum_{j=1}^m \mathbb{E}\Big[\Big|\int_0^T e^{\beta s}\delta Y_s^j\ \mathrm{d}N_s^j\Big|^{p/2}\Big] \le C^* m^{p/2}\ \sum_{j=1}^m \mathbb{E}\Big[\ \int_0^T e^{2\beta s}|\delta Y_s^j|^2\ \mathrm{d}\langle N^j\rangle_s\Big|^{p/4}\Big] \\
&\le C^* m^{p/2}\ \sum_{j=1}^m \mathbb{E}\Big[\Big(\sup_{0\le t\le T} e^{\beta t}|\delta Y_t^j|^2\Big)^{p/4}\ \Big(\int_0^T e^{\beta s}\ \mathrm{d}\langle N^j\rangle_s\Big)^{p/4}\Big] \\
&\le C^* m^{p/2}\ \sum_{j=1}^m \Big(\gamma_2 \mathbb{E}\Big[\Big(\sup_{0\le t\le T} e^{\beta t}|\delta Y_t^j|^2\Big)^{p/2}\Big] + \frac{1}{\gamma_2}\ \mathbb{E}\Big[\Big(\int_0^T e^{\beta s}\ \mathrm{d}\langle N^j\rangle_s\Big)^{p/2}\Big]\Big) \\
&\le C^* m^{p/2}\ \Big(\gamma_2 \|\delta Y\|^p_{\mathcal{S}^p_\beta} + \frac{m}{\gamma_2}\mathbb{E}\Big[\Big(\sum_{j=1}^m \int_0^T e^{\beta s}\ \mathrm{d}\langle N^j\rangle_s\Big)^{p/2}\Big]\Big) \\
&\le C^* m^{p/2+1}\ \left(\gamma_2\|\delta Y\|^p_{\mathcal{S}^p_\beta} + \frac{1}{\gamma_2}\|\delta Z\|^p_{\mathcal{H}^p_\beta}\right) \le d_{p/2}\ \Big\{\gamma_2\|\delta Y\|^p_{\mathcal{S}^p_\beta} + \frac{1}{\gamma_2}\|\delta Z\|^p_{\mathcal{H}^p_\beta}\Big\},
\end{aligned} \tag{5.19}$$

where by (5.14) we have that $C^* m^{p/2+1} = d_{p/2}$. With the particular choice of

$$\gamma_2 := 2^{3p/2-1} d_{p/2} D_2^{-p/2} = 2^{3p/2-1} d_{p/2}\left(\frac{\gamma}{\gamma - \tilde{\alpha}L}\right)^{p/2} > 0,$$

plugging (5.19) into (5.18) yields

$$\begin{aligned}
&\Big(\Big(1 - \frac{\tilde{\alpha}L}{\gamma}\Big)^{p/2} - \frac{2^{3p/2-2}}{\gamma_2} d_{p/2}\Big)\ \|\delta Z\|^p_{\mathcal{H}^p_\beta} = \frac{1}{2} D_2^{p/2}\ \|\delta Z\|^p_{\mathcal{H}^p_\beta} \le \\
&\quad \le 2^{p/2-1}\mathbb{E}\Big[\Big(e^{\beta T}|\delta Y_T|^2\Big)^{p/2}\Big] + 2^{3p/2-2}\mathbb{E}\Big[\Big|\int_0^T e^{\beta s}\big\langle \delta Y_s, \delta_2 f_s\big\rangle \mathrm{d}s\Big|^{p/2}\Big] + 2^{3p/2-2} d_{p/2}\gamma_2\|\delta Y\|^p_{\mathcal{S}^p_\beta},
\end{aligned}$$

which implies the claim.

Step 2: We claim that

$$\begin{aligned}
D_3\|\delta Y\|^p_{\mathcal{S}^p_\beta} \quad &\le \Big(\frac{p}{p-2}\Big)^{p/2}\ \Big\{\Big(2^{p-2} + 2^{3p/2-2}\Big(\frac{\tilde{\alpha}L}{\gamma - \tilde{\alpha}L}\Big)^{p/2}\Big)\mathbb{E}\Big[\Big(e^{\beta T}|\delta Y_T|^2\Big)^{p/2}\Big] \\
&\quad + \Big(2^{3p/2-2} + 2^{5p/2-3}\Big(\frac{\tilde{\alpha}L}{\gamma - \tilde{\alpha}L}\Big)^{p/2}\Big)\ \mathbb{E}\Big[\Big(\int_0^T e^{\beta s}\big|\big\langle \delta Y_s, \delta_2 f_s\big\rangle\big|\mathrm{d}s\Big)^{p/2}\Big]\Big\},
\end{aligned} \tag{5.20}$$

holds for

$$D_3 := 1 - 2^{4p-4} d_{p/2}^2\Big(\frac{p}{p-2}\Big)^{p/2}\Big(\frac{\tilde{\alpha}L}{\gamma - \tilde{\alpha}L}\Big)^{p/2} D_2^{-p/2} - \Big(\frac{\tilde{\alpha}L}{\gamma}T\Big)^{p/2}\Big(\frac{p}{p-2}\Big)^{p/2} 2^{p-2}. \tag{5.21}$$

Note that the choice of K, T and α has been such that $D_3 > 0$ is satisfied. To prove (5.20), we go back to (5.16), where we take the conditional expectation with respect to $\mathcal{F}_t$, then the supremum over $t \in [0, T]$, raise to the power $p/2$ and finally apply Doob's inequality to obtain

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le T}\big(e^{\beta t}|\delta Y_t|^2\big)^{p/2}\Big]\\
&\quad\le \mathbb{E}\Big[\sup_{0\le t\le T}\Big(\mathbb{E}\Big[e^{\beta T}|\delta Y_T|^2+2\int_0^T e^{\beta s}\big|\big\langle \delta Y_s,\delta_2 f_s\big\rangle\big|\mathrm{d}s\\
&\qquad\qquad\qquad + \int_0^T \frac{\tilde\alpha L}{\gamma}e^{\beta s}\big(|\delta Y_s|^2+|\delta Z_s|^2\big)\mathrm{d}s\big|\mathcal{F}_t\Big]\Big)^{p/2}\Big]\\
&\quad\le \big(\frac{p}{p-2}\big)^{p/2}\Big\{2^{p-2}\mathbb{E}\Big[\big(e^{\beta T}|\delta Y_T|^2\big)^{p/2}\Big]+2^{3p/2-2}\mathbb{E}\Big[\big(\int_0^T e^{\beta s}\big|\big\langle \delta Y_s,\delta_2 f_s\big\rangle\big|\mathrm{d}s\big)^{p/2}\Big]\\
&\qquad + 2^{p-2}\mathbb{E}\Big[\big(\int_0^T \frac{\tilde\alpha L}{\gamma}e^{\beta s}|\delta Y_s|^2\mathrm{d}s\big)^{p/2}\Big]+2^{p-2}\mathbb{E}\Big[\big(\int_0^T \frac{\tilde\alpha L}{\gamma}e^{\beta s}|\delta Z_s|^2\mathrm{d}s\big)^{p/2}\Big]\Big\}.
\end{aligned}
\tag{5.22}
$$

Note that we made use of the fact that for $a,b,c,d\in\mathbb{R}$ and $p>2$, we have

$$
\begin{aligned}
\big|a+2b+c+d\big|^{p/2} &\le 2^{p/2-1}\big(|a+2b|^{p/2}+|c+d|^{p/2}\big)\\
&\le 2^{p-2}|a|^{p/2}+2^{3p/2-2}|b|^{p/2}+2^{p-2}|c|^{p/2}+2^{p-2}|d|^{p/2}.
\end{aligned}
$$

Plugging (5.17) into (5.22), we get

$$
\begin{aligned}
\|\delta Y\|^p_{\mathcal{S}^p_\beta} &\le \big(\frac{p}{p-2}\big)^{p/2}\Big\{2^{p-2}\mathbb{E}\Big[\big(e^{\beta T}|\delta Y_T|^2\big)^{p/2}\Big]+2^{3p/2-2}\mathbb{E}\Big[\big(\int_0^T e^{\beta s}\big|\big\langle \delta Y_s,\delta_2 f_s\big\rangle\big|\mathrm{d}s\big)^{p/2}\Big]\\
&\quad + 2^{p-2}\big(\frac{\tilde\alpha L}{\gamma}\big)^{p/2}\|\delta Y\|^p_{\mathcal{H}^p_\beta}+\big(\frac{\tilde\alpha L}{\gamma}\big)^{p/2}\ D_2^{-1}\times 2^{p-2}\Big\{2^{p/2}\mathbb{E}\Big[\big(e^{\beta T}|\delta Y_T|^2\big)^{p/2}\Big]\\
&\quad + 2^{3p/2-1}\mathbb{E}\Big[\big(\int_0^T e^{\beta s}\big|\big\langle \delta Y_s,\delta_2 f_s\big\rangle\big|\mathrm{d}s\big)^{p/2}\Big]+2^{3p-2}d^2_{p/2}D_2^{-p/2}\|\delta Y\|^p_{\mathcal{S}^p_\beta}\Big\}\Big\}\\
&\le \big(\frac{p}{p-2}\big)^{p/2}\Big\{\Big(2^{p-2}+2^{3p/2-2}\big(\frac{\tilde\alpha L}{\gamma-\tilde\alpha L}\big)^{p/2}\Big)\mathbb{E}\Big[\big(e^{\beta T}|\delta Y_T|^2\big)^{p/2}\Big]\\
&\quad + \Big(2^{3p/2-2}+2^{5p/2-3}\big(\frac{\tilde\alpha L}{\gamma-\tilde\alpha L}\big)^{p/2}\Big)\mathbb{E}\Big[\big(\int_0^T e^{\beta s}\big|\big\langle \delta Y_s,\delta_2 f_s\big\rangle\big|\mathrm{d}s\big)^{p/2}\Big]\\
&\quad + \Big(2^{p-2}\big(\frac{\tilde\alpha L}{\gamma}T\big)^{p/2}+2^{4p-4}\big(\frac{\tilde\alpha L}{\gamma-\tilde\alpha L}\big)^{p/2}D_2^{-p/2}d^2_{p/2}\Big)\|\delta Y\|^p_{\mathcal{S}^p_\beta}\Big\},
\end{aligned}
$$

from which the estimate (5.20) follows.

Step 3: At this stage, estimating $\mathbb{E}\Big[\Big(\int_0^T e^{\beta s}\big|\big\langle \delta Y_s,\delta_2 f_s\big\rangle\big|\mathrm{d}s\Big)^{p/2}\Big]$ will yield (5.15). This is a consequence of (5.20): Young's inequality combined with the $\mathcal{S}^p_\beta$-norm yields

$$
\begin{aligned}
\mathbb{E}\Big[\Big(\int_0^T e^{\beta s}\big|\big\langle \delta Y_s,\delta_2 f_s\big\rangle\big|\mathrm{d}s\Big)^{p/2}\Big] &\le \mathbb{E}\Big[\big(\int_0^T e^{\beta s}|\delta Y_s|\ |\delta_2 f_s|\mathrm{d}s\big)^{p/2}\Big]\\
&\le \gamma_3\|\delta Y\|^p_{\mathcal{S}^p_\beta}+\frac{1}{\gamma_3}\mathbb{E}\Big[\Big(\int_0^T e^{\frac{\beta}{2}s}\big|\delta_2 f_s|\mathrm{d}s\Big)^p\Big],
\end{aligned}
\tag{5.23}
$$

and in conjunction with the particular choice

$$\gamma_3 := \frac{1}{2} D_3 \big(\frac{p-2}{p}\big)^{p/2} \frac{(\gamma - \tilde{\alpha} L)^{p/2}}{2^{3p/2-2}(\gamma - \tilde{\alpha} L)^{p/2} + 2^{5p/2-3}(\tilde{\alpha} L)^{p/2}} > 0, \tag{5.24}$$

we get the result. Estimate (5.20) now leads to

$$\begin{aligned} \frac{1}{2} D_3 \|\delta Y\|^p_{\mathcal{S}^p_\beta} \leq \big(\frac{p}{p-2}\big)^{p/2} \Big\{ & \big(2^{p-2} + 2^{3p/2-2}\big(\frac{\tilde{\alpha} L}{\gamma - \tilde{\alpha} L}\big)^{p/2}\big) \mathbb{E}\big[\big(e^{\beta T} |\delta Y_T|^2\big)^{p/2}\big] \\ & + \big(2^{3p/2-2} + 2^{5p/2-3}\big(\frac{\tilde{\alpha} L}{\gamma - \tilde{\alpha} L}\big)^{p/2}\big) \gamma_3^{-1} \ \mathbb{E}\big[\big(\int_0^T e^{\frac{\beta}{2} s} |\delta_2 f_s| \mathrm{d}s\big)^p\big]\Big\}. \end{aligned} \tag{5.25}$$

Note that we trivially have $\|\delta Y\|^p_{\mathcal{H}^p_\beta} \leq T^{p/2} \ \|\delta Y\|^p_{\mathcal{S}^p_\beta}$, hence

$$\|\delta Y\|^p_{\mathcal{S}^p_\beta} + \|\delta Y\|^p_{\mathcal{H}^p_\beta} \leq C^1_p \ \mathbb{E}\big[\big(e^{\beta T} |\delta Y_T|^2\big)^{p/2}\big] + C^2_p \ \mathbb{E}\big[\big(\int_0^T e^{\frac{\beta}{2} s} |\delta_2 f_s| \mathrm{d}s\big)^p\big],$$

where the constants C^1_p and C^2_p are

$$\begin{aligned} C^1_p &:= 2(1 + T^{p/2}) D_3^{-1} \big(\frac{p}{p-2}\big)^{p/2} \big(2^{p-2} + 2^{3p/2-2}\big(\frac{\tilde{\alpha} L}{\gamma - \tilde{\alpha} L}\big)^{p/2}\big), \\ C^2_p &:= 2(1 + T^{p/2}) D_3^{-1} \big(\frac{p}{p-2}\big)^{p/2} \big(2^{3p/2-2} + 2^{5p/2-3}\big(\frac{\tilde{\alpha} L}{\gamma - \tilde{\alpha} L}\big)^{p/2}\big) \gamma_3^{-1}. \end{aligned}$$

Moreover, it follows from (5.17), (5.23) and (5.25) that

$$\|\delta Z\|^p_{\mathcal{H}^p_\beta} \leq C^3_p \ \mathbb{E}\big[\big(e^{\beta T} |\delta Y_T|^2\big)^{p/2}\big] + C^4_p \ \mathbb{E}\big[\big(\int_0^T e^{\frac{\beta}{2} s} |\delta_2 f_s| \mathrm{d}s\big)^p\big].$$

where the constants C^3_p and C^4_p are defined as

$$\begin{aligned} C^3_p :=& 2 D_3^{-1} \big(\frac{p}{p-2}\big)^{p/2} D_2^{-p/2} \Big[2^{p/2} \\ & + \big(2^{3p-2} d^2_{p/2} D_2^{-p/2} + 2^{3p/2-1} \gamma_3\big) \big(2^{p-2} + 2^{3p/2-2}\big(\frac{\tilde{\alpha} L}{\gamma - \tilde{\alpha} L}\big)^{p/2}\big)\Big], \\ C^4_p :=& 2 D_3^{-1} \big(\frac{p}{p-2}\big)^{p/2} D_2^{-p/2} \Big[\big(2^{3p-2} d^2_{p/2} D_2^{-p/2} + 2^{3p/2-1} \gamma_3\big) \times \\ & \times \big(2^{3p/2-2} + 2^{5p/2-3}\big(\frac{\tilde{\alpha} L}{\gamma - \tilde{\alpha} L}\big)\big)^{p/2} \gamma_3^{-1} + 2^{3p/2-1} \gamma_3\Big], \end{aligned}$$

(recall that γ_3 is defined by (5.24)). From the above inequalities we obtain (5.15), where the positive constant C_p is given by

$$C_p := \max\big\{C^1_p + C^3_p, C^2_p + C^4_p\big\}. \tag{5.26}$$

□

Remark 5.2.5. *Note that none of the constants C_p, C_p^i and $D_i, i \in \{1,\dots,4\}$, depend on the terminal condition or $f(\cdot,0,0)$. The only problem related data they do depend on are: K, T, α and the dimension m.*

Remark 5.2.6. *In the previous proof it is clear that our choices for the constants γ_2 and γ_3 do not lead to the most general statement of Proposition 5.2.2. They were chosen this way to avoid a more complex statement, i.e. the constant C_p given in* (5.26) *would then depend on γ_2 and γ_3 and jointly with* (5.13) *we would also have the condition $D_3 > 0$. The conditions of Theorem 5.2.2 below depend on the smallness of C_p as given by* (5.26)*. Our particular choice for γ_2 and γ_3 leads to simpler expressions in our statements.*

Moment estimates - part II

As a by-product of the two previous results we obtain the following moment estimates for the solution of BSDE (5.4).

Corollary 5.2.1 (Moment estimates)**.** *Let $p \geq 2$ and $\beta > 0$. Let $(Y,Z) \in \mathcal{S}^p_\beta \times \mathcal{H}^p_\beta$ be the solution of the delay BSDE* (5.4) *with terminal condition ξ and generator f satisfying (H0)-(H4). For K,T,α small enough, there exists a constant C_p*[1] *such that*

$$\|Y\|^p_{\mathcal{S}^p_\beta} + \|Y\|^p_{\mathcal{H}^p_\beta} + \|Z\|^p_{\mathcal{H}^p_\beta} \leq C_p\Big\{\mathbb{E}\Big[\big(e^{\beta T}|Y_T|^2\big)^{p/2}\Big] + \mathbb{E}\Big[\Big(\int_0^T e^{\beta s}|f(s,0,0)|^2 \mathrm{d}s\Big)^p\Big]\Big\}.$$

The existence and uniqueness result

The moment and a priori estimates in Delong and Imkeller [42] are tailor-made for a Picard iteration in $\mathcal{H}^2 \times \mathcal{H}^2$. To make such a technique work in general L^p spaces we need to state a priori estimates in the form of Proposition 5.2.1 and Proposition 5.2.2. In view of these results one can naturally expect a compatibility condition on K,T and α more complicated than that of Theorem 5.2.1 for a solution to exist.

With estimate (5.15) at hand, we now proceed to show the existence and uniqueness of solutions to (5.4) in $\mathcal{S}^p_\beta \times \mathcal{H}^p_\beta$ for $p > 2$. For $p = 2$, Theorem 2.1 from Delong and Imkeller [42] (reproduced here as Theorem 5.2.1) yields a sufficient condition which guarantees the standard Picard iteration to converge and proves the existence and uniqueness of solutions to (5.4). We show in the following result that for $p > 2$, the convergence of the same Picard iteration is retained. What is needed to achieve this goal is to put some extra effort into proving that the Picard iterates (Y^n, Z^n) satisfy the $\mathcal{S}^p_\beta, \mathcal{H}^p_\beta$-integrability properties.

Theorem 5.2.2. *Let $p > 2$ and assume that (H0)-(H4) hold. Let K or T or α be small enough such that for some $\beta, \gamma > 0$ the conditions of Proposition 5.2.2 are satisfied. If further K or T or α are small enough such that we have*

$$2^{p/2-1} C_p \Big(LT \int_{-T}^0 e^{-\beta s}\rho(\mathrm{d}s)\Big)^{p/2} \max\{1, T^{p/2}\} < 1, \quad \textit{for } \rho \in \{\alpha_{\mathcal{Y}}, \alpha_{\mathcal{Z}}\}, \tag{5.27}$$

[1]As in Propositions 5.2.1 and 5.2.2, C_p depends on several constants that can be suitably chosen.

where $C_p = C_p(\beta, \gamma, \tilde{\alpha}, L, T, m) > 0$ *is given by* (5.26), $\tilde{\alpha}$ *is given by* (5.7) *and* $L = K\alpha$, *then the BSDE* (5.4) *admits a unique solution* (Y, Z) *in* $\mathcal{S}^p_\beta \times \mathcal{H}^p_\beta$.

Remark 5.2.7. *Denote by* $J := TK\alpha$ *the product of* T, K *and* α. *Note that, by definition of the constant* C_p, *condition* (5.27) *is satisfied if either* T *or* K *or* α *is small enough since* $\lim_{J \to 0} C_p < +\infty$ *which in turn implies*

$$\lim_{J \to 0} C_p (\alpha K T)^{p/2} = 0.$$

Proof of Theorem 5.2.2. Let $p > 2$. Let $t \in [0, T]$. The proof is based on the standard Picard iteration: we initialize by $Y^0 = 0$ and $Z^0 = 0$ and define recursively

$$Y^{n+1}_t = \xi + \int_t^T f\big(s, \Gamma^n(s)\big)\mathrm{d}s - \int_t^T Z^{n+1}_s \mathrm{d}W_s, \quad 0 \le t \le T, \tag{5.28}$$

with $\Gamma^n(s) = \big(\int_{-T}^0 Y^n_{s+v}\alpha_{\mathcal{Y}}(\mathrm{d}v), \int_{-T}^0 Z^n_{s+v}\alpha_{\mathcal{Z}}(\mathrm{d}v)\big)$ for $s \in [0, T]$ and $n \in \mathbb{N}$. In the following, let $C > 0$ denote some generic constant which may vary from line to line but is always independent of $n \in \mathbb{N}$. We proceed by induction, where the existence of $(Y^1, Z^1) \in \mathcal{S}^p_\beta \times \mathcal{H}^p_\beta$ follows from classical stochastic analysis arguments. For $n \ge 1$, assume that $(Y^n, Z^n) \in \mathcal{S}^p_\beta \times \mathcal{H}^p_\beta$ solves the BSDE (5.28) and we now prove that (5.28) has a unique solution $(Y^{n+1}, Z^{n+1}) \in \mathcal{S}^p_\beta \times \mathcal{H}^p_\beta$. Note that due to

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^T |f(s, \Gamma^n(s))|\mathrm{d}s\Big)^p\Big] \\
&\quad \le \mathbb{E}\Big[\Big(\int_0^T |f(s,0,0)|\mathrm{d}s + \int_0^T |f(s,\Gamma^n(s)) - f(s,0,0)|\mathrm{d}s\Big)^p\Big] \\
&\quad \le 2^{p-1}\, \mathbb{E}\Big[\Big(\int_0^T |f(s,0,0)|\mathrm{d}s\Big)^p + \Big(T\int_0^T |f(s,\Gamma^n(s)) - f(s,0,0)|^2\mathrm{d}s\Big)^{p/2}\Big] \\
&\quad \le 2^{p-1}\, \mathbb{E}\Big[\Big(\int_0^T |f(s,0,0)|\mathrm{d}s\Big)^p \\
&\qquad + L^{p/2}T^{p/2}\Big\{\int_0^T \int_{-T}^0 |Y^n_{s+v}|^2\alpha_{\mathcal{Y}}(\mathrm{d}v)\mathrm{d}s + \int_0^T\int_{-T}^0 |Z^n_{s+v}|^2\alpha_{\mathcal{Z}}(\mathrm{d}v)\mathrm{d}s\Big\}^{p/2}\Big] \\
&\quad \le 2^{p-1}\mathbb{E}\Big[\Big(\int_0^T |f(s,0,0)|\mathrm{d}s\Big)^p + (\alpha K T)^{p/2}\Big\{\int_0^T |Y^n_s|^2\mathrm{d}s + \int_0^T |Z^n_s|^2\mathrm{d}s\Big\}^{p/2}\Big] \\
&\quad \le 2^{p-1}\mathbb{E}\Big[\Big(\int_0^T |f(s,0,0)|\mathrm{d}s\Big)^p\Big] + 2^{p/2-1}(2\alpha K T)^{p/2}\Big(T^{p/2}\|Y^n\|^p_{\mathcal{S}^p_0} + \|Z^n\|^p_{\mathcal{H}^p_0}\Big) < \infty,
\end{aligned} \tag{5.29}$$

the martingale representation yields a uniquely determined process $Z^{n+1} \in \mathcal{H}^2_0$ such that

$$\mathbb{E}\Big[\xi + \int_0^T f\big(s, \Gamma^n(s)\big)\mathrm{d}s \big| \mathcal{F}_t\Big] = \mathbb{E}\Big[\xi + \int_0^T f\big(s, \Gamma^n(s)\big)\mathrm{d}s\Big] + \int_0^t Z^{n+1}_s \mathrm{d}W_s,\ t \in [0, T].$$

We then define Y^{n+1} to be a continuous version of $Y^{n+1}_t = \mathbb{E}[\xi + \int_t^T f(s, \Gamma^n(s))\mathrm{d}s | \mathcal{F}_t]$.

Let us first show that $Y^{n+1} \in \mathcal{S}^p_0$:

$$\|Y^{n+1}\|^p_{\mathcal{S}^p_0} = \mathbb{E}\Big[\sup_{t\in[0,T]} |Y^{n+1}_t|^p\Big] \leq \mathbb{E}\Big[\sup_{t\in[0,T]} \Big(\mathbb{E}\big[\,|\xi| + \int_0^T |f(s,\Gamma^n(s))|\mathrm{d}s\,|\mathcal{F}_t\big]\Big)^p\Big]$$
$$\leq \Big(\frac{p}{p-1}\Big)^p \mathbb{E}\Big[\Big(|\xi| + \int_0^T |f(s,\Gamma^n(s))|\mathrm{d}s\Big)^p\Big]$$
$$\leq 2^{p-1}\Big(\frac{p}{p-1}\Big)^p\ \mathbb{E}\Big[|\xi|^p + \Big(\int_0^T |f(s,\Gamma^n(s))|\mathrm{d}s\Big)^p\Big] < \infty,$$

where the last inequality follows from the fact that $\xi \in L^p$ and (5.29). This proves that $Y^{n+1} \in \mathcal{S}^p_0$. Since all $\|\cdot\|_{\mathcal{S}^p_\beta}$-norms are equivalent it follows that $Y^{n+1} \in \mathcal{S}^p_\beta$. To see that $Z^{n+1} \in \mathcal{H}^p_\beta$, recall that Itô's formula applied to $e^{\beta t}|Y^{n+1}_t|^2$ yields

$$e^{\beta t}|Y^{n+1}_t|^2 + \int_t^T \beta e^{\beta s}|Y^{n+1}_s|^2\mathrm{d}s + \int_t^T e^{\beta s}|Z^{n+1}_s|^2\mathrm{d}s$$
$$= e^{\beta T}|\xi|^2 + \int_t^T 2e^{\beta s}\langle Y^{n+1}_s, f(s,\Gamma^n(s))\rangle\mathrm{d}s - \int_t^T 2e^{\beta s}\langle Y^{n+1}_s, Z^{n+1}_s\mathrm{d}W_s\rangle.$$

In the above drop the two Y terms on the left-hand side of the equation, take $t = 0$, apply absolute values to both sides and then raise to power $p/2$. This yields

$$\Big(\int_0^T e^{\beta s}|Z^{n+1}_s|^2\mathrm{d}s\Big)^{p/2}$$
$$\leq \Big(e^{\beta T}|\xi|^2 + \int_0^T 2e^{\beta s}|Y^{n+1}_s|\,|f(s,\Gamma^n(s))|\mathrm{d}s + \Big|\int_0^T 2e^{\beta s}\langle Y^{n+1}_s, Z^{n+1}_s\mathrm{d}W_s\rangle\Big|\Big)^{p/2}$$
$$\leq\ 2^{p/2-1}\big(e^{\beta T}|\xi|^2\big)^{p/2} + 2^{p-2}\Big(\int_0^T 2e^{\beta s}|Y^{n+1}_s|\,|f(s,\Gamma^n(s))|\mathrm{d}s\Big)^{p/2}$$
$$+\, 2^{3p/2-2}\Big|\int_0^T e^{\beta s}\langle Y^{n+1}_s, Z^{n+1}_s\mathrm{d}W_s\rangle\Big|^{p/2}. \tag{5.30}$$

On the one hand, we have

$$\mathbb{E}\Big[\Big(\int_0^T 2e^{\beta s}|Y^{n+1}_s|\,|f(s,\Gamma^n(s))|\mathrm{d}s\Big)^{p/2}\Big]$$
$$\leq \mathbb{E}\Big[\Big(\int_0^T 2e^{\beta s}|Y^{n+1}_s|\,|f(s,\Gamma^n(s)) - f(s,0,0)|\mathrm{d}s + \int_0^T 2e^{\beta s}|Y^{n+1}_s|\,|f(s,0,0)|\mathrm{d}s\Big)^{p/2}\Big]$$
$$\leq C\Big\{\|Y^{n+1}\|^p_{\mathcal{S}^p_\beta} + \mathbb{E}\Big[\Big(\int_0^T e^{\frac{\beta}{2}s}|f(s,0,0)|\mathrm{d}s\Big)^p\Big] + \|Y^n\|^p_{\mathcal{S}^p_\beta} + \|Z^n\|^p_{\mathcal{H}^p_\beta}\Big\} < \infty, \tag{5.31}$$

where we have used the Lipschitz condition of f in combination with a similar calculation as in (5.29) and the estimate

$$\int_0^T 2e^{\beta s}|Y^{n+1}_s|\,|f(s,0,0)|\mathrm{d}s \leq \sup_{0\leq t\leq T} e^{\beta t}|Y^{n+1}_t|^2 + \Big(\int_0^T e^{\frac{\beta}{2}s}|f(s,0,0)|\mathrm{d}s\Big)^2.$$

On the other hand, by the same arguments as in (5.19) we find the estimate

$$\mathbb{E}\Big[\Big|\int_0^T e^{\beta s}\langle Y_s^{n+1}, Z_s^{n+1}\mathrm{d}W_s\rangle\Big|^{p/2}\Big] \le d_{p/2}\ \Big\{\kappa\ \|Y^{n+1}\|^p_{\mathcal{S}^p_\beta} + \frac{1}{\kappa}\|Z^{n+1}\|^p_{\mathcal{H}^p_\beta}\Big\}, \tag{5.32}$$

where in the last line the constant $\kappa > 0$ appears due to Young's inequality. Now choosing $\kappa > 0$ such that $1 - 2^{2p-2}\ d_{p/2}\kappa^{-1} > 0$, it follows from (5.30), (5.31) and (5.32) that

$$\begin{aligned}\Big(1 - \frac{2^{2p-2}\ d_{p/2}}{\kappa}\Big)\|Z^{n+1}\|^p_{\mathcal{H}^p_\beta} \le C\Big\{&\mathbb{E}\Big[\big(e^{\beta T}|\xi|^2\big)^{p/2}\Big] + \|Y^{n+1}\|^p_{\mathcal{S}^p_\beta} \\ &+ \mathbb{E}\Big[\Big(\int_0^T |f(s,0,0)|\mathrm{d}s\Big)^p\Big] + \|Y^n\|^p_{\mathcal{S}^p_\beta} + \|Z^n\|^p_{\mathcal{H}^p_\beta}\Big\} < \infty.\end{aligned}$$

This proves that $Z^{n+1} \in \mathcal{H}^p_\beta$.

In the next step, we prove that the sequence (Y^n, Z^n) converges in $\mathcal{S}^p_\beta \times \mathcal{H}^p_\beta$. Under the current assumptions one is able to apply a priori estimate (5.15) to obtain

$$\begin{aligned}&\|Y^{n+1} - Y^n\|^p_{\mathcal{S}^p_\beta} + \|Z^{n+1} - Z^n\|^p_{\mathcal{H}^p_\beta} \\ &\le C_p\ \mathbb{E}\Big[\Big(\int_0^T e^{\frac{\beta}{2}s}\big|f(s,\Gamma^n(s)) - f(s,\Gamma^{n-1}(s))\big|\mathrm{d}s\Big)^p\Big] \\ &\le C_p T^{p/2}\ \mathbb{E}\Big[\Big(\int_0^T e^{\beta s}\big|f(s,\Gamma^n(s)) - f(s,\Gamma^{n-1}(s))\big|^2\mathrm{d}s\Big)^{p/2}\Big].\end{aligned}$$

In analogy to the calculation carried out in Equation (2.7) in Delong and Imkeller [42][Proof of Theorem 2.1], it is straightforward to see that we have

$$\begin{aligned}&\|Y^{n+1} - Y^n\|^p_{\mathcal{S}^p_\beta} + \|Z^{n+1} - Z^n\|^p_{\mathcal{H}^p_\beta} \\ &\le C_p T^{p/2}\ \mathbb{E}\Big[\Big(L\max\Big\{\int_{-T}^0 e^{-\beta s}\alpha_{\mathcal{Y}}(\mathrm{d}s), \int_{-T}^0 e^{-\beta s}\alpha_{\mathcal{Z}}(\mathrm{d}s)\Big\} \\ &\qquad \times \Big(T\sup_{t\in[0,T]} e^{\beta t}|Y_t^n - Y_t^{n-1}|^2 + \int_0^T e^{\beta s}|Z_s^n - Z_s^{n-1}|^2\mathrm{d}s\Big)\Big)^{p/2}\Big] \\ &\le C_p T^{p/2}\ 2^{p/2-1}\ \Big(L\max\Big\{\int_{-T}^0 e^{-\beta s}\alpha_{\mathcal{Y}}(\mathrm{d}s), \int_{-T}^0 e^{-\beta s}\alpha_{\mathcal{Z}}(\mathrm{d}s)\Big\}\Big)^{p/2} \\ &\qquad \times \Big(T^{p/2}\|Y^n - Y^{n-1}\|^p_{\mathcal{S}^p_\beta} + \|Z^n - Z^{n-1}\|^p_{\mathcal{H}^p_\beta}\Big) \\ &\le C_p\ 2^{p/2-1}\ \Big(LT\max\Big\{\int_{-T}^0 e^{-\beta s}\alpha_{\mathcal{Y}}(\mathrm{d}s), \int_{-T}^0 e^{-\beta s}\alpha_{\mathcal{Z}}(\mathrm{d}s)\Big\}\Big)^{p/2}\ \max\big\{1, T^{p/2}\big\} \\ &\qquad \times \Big(\|Y^n - Y^{n-1}\|^p_{\mathcal{S}^p_\beta} + \|Z^n - Z^{n-1}\|^p_{\mathcal{H}^p_\beta}\Big).\end{aligned}$$

Hence, by (5.27), the standard fixed point argument yields that (Y^n, Z^n) converges in $\mathcal{S}^p_\beta \times \mathcal{H}^p_\beta$, which finishes the proof. □

5.3 Decoupled FBSDE with time delayed generators

The objective of this section is to extend the results from Delong and Imkeller [42, 43] to the case of decoupled forward-backward stochastic differential equations. For measurable functions b, σ, g, f, specified in more detail below, we study the time delayed FBSDE

$$X_t^x = x + \int_0^t b(s, X_s^x)\mathrm{d}s + \int_0^t \sigma(s, X_s^x)\mathrm{d}W_s, \quad x \in \mathbb{R}^d, \tag{5.33}$$

$$Y_t^x = g(X_T^x) + \int_t^T f\big(s, \Theta^x(s)\big)\mathrm{d}s - \int_t^T Z_s^x \mathrm{d}W_s, \quad 0 \le t \le T, \tag{5.34}$$

where for $t \in [0, T]$, we write

$$\begin{aligned}\Theta^x(t) &= \big((X^x \cdot \alpha_{\mathcal{X}})(t), (Y^x \cdot \alpha_{\mathcal{Y}})(t), (Z^x \cdot \alpha_{\mathcal{Z}})(t)\big) \\ &= \Big(\int_{-T}^0 X_{t+v}^x \alpha_{\mathcal{X}}(\mathrm{d}v), \int_{-T}^0 Y_{t+v}^x \alpha_{\mathcal{Y}}(\mathrm{d}v), \int_{-T}^0 Z_{t+v}^x \alpha_{\mathcal{Z}}(\mathrm{d}v)\Big),\end{aligned} \tag{5.35}$$

with given deterministic finite measures $\alpha_{\mathcal{X}}, \alpha_{\mathcal{Y}}$ and $\alpha_{\mathcal{Z}}$ supported on $[-T, 0)$. The coefficients b, σ, g, f appearing in (5.33)-(5.34) are assumed to satisfy certain smoothness and integrability conditions such that the backward equation (5.34) falls back into the setting of (H0)-(H4) from Section 5.2.1. More precisely, we assume the following to hold:

(F0) $\alpha_{\mathcal{Y}}$, $\alpha_{\mathcal{Y}}, \alpha_{\mathcal{Z}}$ are three non-random, finitely valued measures supported on $[-T, 0)$;

(F1) $g : \mathbb{R}^d \to \mathbb{R}^m$ is continuous differentiable with uniformly bounded first order derivatives, i.e. there exists $K' > 0$ such that $|\nabla g| \le K'$;

(F2) $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R}^m$ is continuously differentiable with uniformly bounded derivatives, i.e. there exists a constant $K > 0$ such that[2]

$$|\nabla_x f|,\ |\nabla_y f|,\ |\nabla_z f| \le \sqrt{K/3}$$

holds uniformly in all variables; f satisfies a uniform Lipschitz condition with Lipschitz constant $\sqrt{K/3}$.

(F3) $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ are continuously differentiable functions with bounded derivatives; $|b(\cdot, 0)|$ and $|\sigma(\cdot, 0)|$ are uniformly bounded; σ is elliptic;

(F4) $\big(\int_0^T |f(s, 0, 0, 0)|^2 \mathrm{d}s\big)^{p/2} < \infty$ for $p \ge 2$;

(F5) $f(t, \cdot, \cdot, \cdot)\mathbb{1}_{(-\infty,0)}(t) = 0$;

[2]We remark that this bound is taken over the corresponding Euclidean norm of the derivative matrix/tensor. To avoid possible confusion when using tensors one can always interpret the variable $z \in \mathbb{R}^{m \times d}$ of the mapping f as a sequence of d-dimensional vectors $z_i \in \mathbb{R}^d$ ($i \in \{1, \cdots, m\}$) rather than a matrix. The condition would then read $\sum_{i=1}^m |\nabla_{z_i} f| \le \sqrt{K/3}$ where $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^m \times \underbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}_{m-\text{times}} \to \mathbb{R}^m$.

Condition (F3) is a standard assumption which guarantees the existence and uniqueness of the solution of SDE (5.33). Furthermore, condition (F2) implies that the generator is uniformly Lipschitz continuous in $(x,y,z) \in \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$. In analogy to conditions (H2) and (H2') from Section 5.2.1, let us spell out the following implication of the Lipschitz condition (F2): with the constant $K > 0$ chosen above, for any $t \in [0,T]$ and any sufficiently integrable vector or matrix valued processes u, u', y, y' and z, z', we have

$$\text{(F2')}\quad \begin{aligned} &\Big| f\big(t,(u\cdot\alpha_{\mathcal{X}})(t),(y\cdot\alpha_{\mathcal{Y}})(t),(z\cdot\alpha_{\mathcal{Z}})(t)\big) \\ &\qquad\qquad - f\big(t,(u'\cdot\alpha_{\mathcal{X}})(t),(y'\cdot\alpha_{\mathcal{Y}})(t),(z'\cdot\alpha_{\mathcal{Z}})(t)\big)\Big|^2 \\ &\le K\Big(\big|(u\cdot\alpha_{\mathcal{X}})(t)-(u'\cdot\alpha_{\mathcal{X}})(t)\big|^2 \\ &\qquad\qquad + \big|(y\cdot\alpha_{\mathcal{Y}})(t)-(y'\cdot\alpha_{\mathcal{Y}})(t)\big|^2 + \big|(z\cdot\alpha_{\mathcal{Z}})(t)-(z'\cdot\alpha_{\mathcal{Z}})(t)\big|^2\Big) \\ &\le K\alpha_{\mathcal{X}}([-T,0])\big((x-x')^2\cdot\alpha_{\mathcal{X}}\big)(t) + L\Big(\big((y-y')^2\cdot\alpha_{\mathcal{Y}}\big)(t) + \big((z-z')^2\cdot\alpha_{\mathcal{Z}}\big)(t)\Big) \end{aligned}$$

where $L = K\alpha$ with α defined in (5.3). For a fixed $x \in \mathbb{R}^d$, the existence and uniqueness of solutions to the backward equation (5.34) in $\mathcal{S}^2_\beta \times \mathcal{H}^2_\beta$ is guaranteed under the assumptions (F0)-(F5) together with the compatibility criterion from Theorem 5.2.1 on the terminal time and the Lipschitz constant $L = K\alpha$, i.e.

$$\big(8T + \frac{1}{\beta}\big)L\int_{-T}^0 e^{-\beta s}\rho(\mathrm{d}s)\max\{1,T\} < 1, \quad \text{for } \rho \in \{\alpha_{\mathcal{Y}}, \alpha_{\mathcal{Z}}\}.$$

To extend the result to $\mathcal{S}^p_\beta \times \mathcal{H}^p_\beta$ for $p > 2$, one only needs to replace the condition above by the compatibility condition from Theorem 5.2.2,

$$2^{p/2-1}C_p\Big(LT\int_{-T}^0 e^{-\beta s}\rho(\mathrm{d}s)\Big)^{p/2}\max\{1,T^{p/2}\} < 1, \quad \text{for } \rho \in \{\alpha_{\mathcal{Y}}, \alpha_{\mathcal{Z}}\}.$$

Throughout this section, given $p \ge 2$, we will assume that for every $x \in \mathbb{R}^d$, the FBSDE (5.33)-(5.34) admits a unique solution $(X^x, Y^x, Z^x) \in \mathcal{S}^q_\beta(\mathbb{R}^d) \times \mathcal{S}^p_\beta(\mathbb{R}^m) \times \mathcal{H}^p_\beta(\mathbb{R}^{m\times d})$ for all $q \ge 2$.

5.3.1 Gâteaux and Norm differentiability

In this section we investigate the variational differentiability of the solution (X^x, Y^x, Z^x) of the time delayed FBSDE (5.33)-(5.34) with respect to the Euclidean parameter $x \in \mathbb{R}^d$, i.e. with respect to the initial condition of the forward diffusion. By a well known result (see e.g. Protter [114]), (F3) implies that the forward component X^x is differentiable with respect to the parameter $x \in \mathbb{R}^d$. It is natural to pose the question whether this smoothness is carried over to (Y^x, Z^x) in the setting of FBSDE with time delayed generators. In this section we denote by h an element of $\mathbb{R}^d \setminus \{0\}$. Our goal is to show that the variational equations of (5.33)-(5.34) are given by

$$\nabla X_t^x h = h + \int_0^t \nabla b(s, X_s^x) \nabla X_s^x h \, \mathrm{d}s + \int_0^t \nabla \sigma(s, X_s^x) \nabla X_s h \, \mathrm{d}W_s, \tag{5.36}$$

$$\nabla Y_t^x h = \nabla g(X_T^x) \nabla X_T^x h - \int_t^T \nabla Z_s^x h \, \mathrm{d}W_s + \int_t^T \big\langle (\nabla f)\big(s, \Theta^x(s)\big), (\nabla \Theta^x h)(s) \big\rangle \mathrm{d}s, \tag{5.37}$$

where the notation ∇X^x (respectively ∇Y^x and ∇Z^x) denotes the Gâteaux derivatives of X^x (respectively Y^x and Z^x) in the direction h and $(\nabla \Theta^x h)(t)$ is to be understood in the same fashion as in (5.35), i.e.

$$(\nabla \Theta^x h)(t) = \big((\nabla X^x h \cdot \alpha_{\mathcal{X}})(t), (\nabla Y^x h \cdot \alpha_{\mathcal{Y}})(t), (\nabla Z^x h \cdot \alpha_{\mathcal{Z}})(t) \big), \quad t \in [0, T]. \tag{5.38}$$

Note that (F3) implies that (5.36) admits a unique solution in $\mathcal{S}_\beta^p$ for every $\beta \geq 0$ and $p \geq 2$. Let (X, Y, Z) and $\nabla X h$ solve (5.33)-(5.34) and (5.36) respectively and let Θ^x be as defined by (5.35). Now consider the BSDE with the linear time delayed generator for $t \in [0, T]$

$$P_t h = \nabla g(X_T^x) \nabla X_T^x h - \int_t^T Q_s h \, \mathrm{d}W_s + \int_t^T \widehat{F}\big(s, (Ph \cdot \alpha_{\mathcal{Y}})(s), (Qh \cdot \alpha_{\mathcal{Z}})(s)\big) \mathrm{d}s, \tag{5.39}$$

where $\widehat{F} : \Omega \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R}^m$ and $\widehat{F}(t, p, q) = \langle (\nabla f)\big(t, \Theta^x(t)\big), \big((\nabla X^x h \cdot \alpha_{\mathcal{X}})(t), p, q\big) \rangle$. The next corollary states, by means of Theorem 5.2.1 and Proposition 5.2.1, a result concerning the existence and uniqueness of solution to (5.39). This solution process will then serve as the natural candidate (in a yet to be specified sense) for $\nabla_x Y^x h$ and $\nabla_x Z^x h$, solution to (5.37).

Corollary 5.3.1. *Let $p \geq 2$, $h \in \mathbb{R}^d \setminus \{0\}$ and $\beta > 0$. Assume that (F0)-(F5) are satisfied and let $L > 0$ be as in (F2'). If $p > 2$ assume that T, K, α are chosen like in Proposition 5.2.2 and satisfy in addition*

$$2^{p/2-1} C_p \Big(LT \int_{-T}^0 e^{-\beta s} \rho(\mathrm{d}s) \Big)^{p/2} \max\{1, T^{p/2}\} < 1, \quad \textit{for } \rho \in \{\alpha_{\mathcal{Y}}, \alpha_{\mathcal{Z}}\}, \tag{5.40}$$

If $p = 2$ assume T, K, α are chosen such that the conditions of Theorem 5.2.1 and of Proposition 5.2.1 hold. Then for every fixed x in $\mathbb{R}^d$, BSDE (5.34) has a unique solution $(Y, Z) \in \mathcal{S}_\beta^p \times \mathcal{H}_\beta^p$ and BSDE (5.39) has a unique solution $(Ph, Qh) \in \mathcal{S}_\beta^p \times \mathcal{H}_\beta^p$.

Proof. Given the known properties of X and ∇X (and hence of $\nabla X h$) it is easy to see that $\xi = \nabla g(X_T^x) \nabla X_T^x h$ and $\widehat{F}(\cdot, 0, 0)$ satisfy conditions (H1), (H3) and (H4). We recall that by Remark 5.2.5, the several compatibility conditions (5.40) as well as the conditions in Proposition 5.2.2 depend only on the Lipschitz constant K of (F2), the delay measures $\alpha_{\mathcal{Y}}$, $\alpha_{\mathcal{Z}}$, T and the dimension of the equations.

From the definition of $\widehat{F}$ and by the bounds of the (spatial) derivatives of f assumed in (F2) it is clear that $\widehat{F}$ satisfies a standard Lipschitz condition (in the spatial variables).

In particular, take $p, p' \in \mathbb{R}^m$ and[3] $q, q' \in \mathbb{R}^{m\times d}$, then via the Minkowski and the Cauchy-Schwarz inequality together with (F2) we have

$$
\begin{aligned}
|\widehat{F}(t,p,q) - \widehat{F}(t,p',q')| &\le \big|\langle (\nabla_y f)\big(t,\Theta^x(t)\big),(p-p')\rangle\big| + \big|\langle (\nabla_z f)\big(t,\Theta^x(t)\big),(q-q')\rangle\big| \\
&\le |(\nabla_y f)|\,|p-p'| + |(\nabla_z f)|\,|q-q'| \le \sqrt{K/3}\big(\,|p-p'| + |q-q'|\,\big).
\end{aligned}
$$

Hence $\widehat{F}$ satisfies exactly the same Lipschitz condition as f. Furthermore, the delay measures appearing in $\widehat{F}$ are the same as those that appear in f. We can thus conclude that the Lipschitz constant, the delay measures, terminal time T and dimensions for f and $\widehat{F}$ are the same. Under this corollary's assumptions, the conditions of Theorem 5.2.2 are satisfied for both BSDEs (5.34) and (5.39). The existence of a unique solution (Y,Z) and (Ph,Qh) in $\mathcal{S}^p_\beta \times \mathcal{H}^p_\beta$ of (5.34) and (5.39) (respectively) follow from Theorem 5.2.2 and Theorem 5.2.1. □

The solution of BSDE (5.39) serves now as the natural candidate for the variational derivatives of (Y,Z), solution of (5.37). If one shows that $(\nabla Y^x h, \nabla Z^x h)$ exist in some sense then by the uniqueness of the solution of (5.39), the solutions to (5.37) and (5.39) must coincide, i.e. $\big(\nabla Y^x h, \nabla Z^x h\big) = \big(Ph, Qh\big)$ must hold almost surely.

For the rest of the section, we assume that all assumptions ensuring the existence and uniqueness of the variational equations (5.36)-(5.37) are fulfilled, i.e. we assume that the assumptions of Corollary 5.3.1 hold. In our next result we show that the mapping $x \mapsto (Y^x, Z^x)$ is differentiable in an adequate sense.

Proposition 5.3.1. *Let $p \ge 2$ and assume that the conditions of Corollary (5.3.1) hold. Then, for every $x \in \mathbb{R}^d$ the solution (X^x, Y^x, Z^x) of the FBSDE (5.33)-(5.34) is norm-differentiable in the sense*

$$
\lim_{\varepsilon\to 0}\left\| \frac{Y^{x+\varepsilon h} - Y^x}{\varepsilon} - \nabla Y^x h \right\|^p_{\mathcal{S}^p_\beta} = \lim_{\varepsilon\to 0}\left\| \frac{Z^{x+\varepsilon h} - Z^x}{\varepsilon} - \nabla Z^x h \right\|^p_{\mathcal{H}^p_\beta} = 0, \quad \forall h \in \mathbb{R}^d \setminus \{0\},
$$

where $(\nabla Y^x h, \nabla Z^x h)$ is the unique solution of the BSDE

$$
\nabla Y^x_t h = \nabla g(X^x_T)\nabla X^x_T h - \int_t^T \nabla Z^x_s h \,\mathrm{d}W_s + \int_t^T \big\langle (\nabla f)\big(s,\Theta^x(s)\big), (\nabla\Theta^x h)(s)\big\rangle \mathrm{d}s,
$$

with Θ^x and $\nabla\Theta^x$ defined by (5.35) and (5.38) respectively.

[3] Or a sequence of $q_i, q_i' \in \mathbb{R}^m$ with $i \in \{1,\cdots,d\}$ as we saw in page 122's footnote.

Proof. Let $x \in \mathbb{R}^d$, $t \in [0,T]$ and $\varepsilon > 0$. We use the following notations:

$$\begin{aligned}
A_{s,\mathcal{X}} &:= \int_0^1 \nabla_x f\Big(s, (X^x \cdot \alpha_{\mathcal{X}})(s) + \theta\big((X^{x+\varepsilon h} - X^x) \cdot \alpha_{\mathcal{X}}\big)(s), \\
&\qquad\qquad (Y^{x+\varepsilon h} \cdot \alpha_{\mathcal{Y}})(s), (Z^{x+\varepsilon h} \cdot \alpha_{\mathcal{Z}})(s)\Big)\mathrm{d}\theta, \\
A_{s,\mathcal{Y}} &:= \int_0^1 \nabla_y f\Big(s, (X^x \cdot \alpha_{\mathcal{X}})(s), \\
&\qquad\qquad (Y^x \cdot \alpha_{\mathcal{Y}})(s) + \theta\big((Y^{x+\varepsilon h} - Y^x) \cdot \alpha_{\mathcal{Y}}\big)(s), (Z^{x+\varepsilon h} \cdot \alpha_{\mathcal{Z}})(s)\Big)\mathrm{d}\theta, \\
A_{s,\mathcal{Z}} &:= \int_0^1 \nabla_z f\Big(s, (X^x \cdot \alpha_{\mathcal{X}})(s), \\
&\qquad\qquad (Y^x \cdot \alpha_{\mathcal{Y}})(s), (Z^x \cdot \alpha_{\mathcal{Z}})(s) + \theta\big((Z^{x+\varepsilon h} - Z^x) \cdot \alpha_{\mathcal{Z}}\big)(s)\Big)\mathrm{d}\theta.
\end{aligned} \tag{5.41}$$

We point out that though the processes A depend on ε and x, for the sake of notational simplicity we do not spell out this dependence explicitly. We furthermore remark that by assumption (F2), we have $|A_{\cdot,*}| \le \sqrt{K/3}$ for $* = \mathcal{X}, \mathcal{Y}, \mathcal{Z}$, so in particular they are uniformly bounded in x and ε.

Let us denote by (Ph, Qh) the solution of the BSDE (5.39) which coincides with $(\nabla Y h, \nabla Z h)$. We furthermore define the auxiliary processes $\xi := \big(g(X_T^{x+\varepsilon h}) - g(X_T^x)\big)/\varepsilon - \nabla g(X_T^x)\nabla X_T^x h$ and

$$U := \frac{Y^{x+\varepsilon h} - Y^x}{\varepsilon} - Ph, \quad V := \frac{Z^{x+\varepsilon h} - Z^x}{\varepsilon} - Qh, \quad \text{and} \quad \tilde{X} := \frac{X^{x+\varepsilon h} - X^x}{\varepsilon} - \nabla X^x h. \tag{5.42}$$

Note that from (F2) and standard results on SDEs we have that $\tilde{X}$ is well-defined and $\tilde{X} \in \mathcal{S}_\beta^p$ for any $\beta \ge 0$ and $p \ge 2$. We now claim

$$\lim_{\varepsilon \to 0} \|U\|_{\mathcal{S}_\beta^p}^p = \lim_{\varepsilon \to 0} \|V\|_{\mathcal{H}_\beta^p}^p = 0, \quad \text{for arbitrary } x \in \mathbb{R}^d.$$

This result obviously proves the norm differentiability. To start with, we have

$$\begin{aligned}
U_t = \xi &+ \int_t^T \frac{f(s, \Theta^{x+\varepsilon h}(s)) - f(s, \Theta^x(s))}{\varepsilon}\mathrm{d}s \\
&- \int_t^T \Big\langle (\nabla f)\big(s, \Theta^x(s)\big), \big((\nabla X^x h \cdot \alpha_{\mathcal{X}})(s), (Ph \cdot \alpha_{\mathcal{Y}})(s), (Qh \cdot \alpha_{\mathcal{Z}})(s)\big)\Big\rangle \mathrm{d}s \\
&- \int_t^T V_s \mathrm{d}W_s.
\end{aligned}$$

By construction the above equation is well-defined because for every $x \in \mathbb{R}^d$ and $\varepsilon > 0$, all the involved processes are known a priori to exist and have the necessary integrability properties. The format of the above dynamics is still not convenient, so we transform it into the more familiar dynamics of a delay BSDE. Using the identity $\phi(x) - \phi(y) = (x - y)\int_0^1 \nabla\phi(y + \theta(x - y))\mathrm{d}\theta$ for a continuously differentiable function $\phi : \mathbb{R}^a \to \mathbb{R}^b$ (a

and b being arbitrary non-zero integers), the previous equation can be rewritten as

$$
\begin{aligned}
U_t &= \xi + \frac{1}{\varepsilon}\int_t^T \Big[A_{s,\mathcal{X}}\big((X^{x+\varepsilon h} - X^x)\cdot\alpha_{\mathcal{X}}\big)(s) \\
&\quad + A_{s,\mathcal{Y}}\big((Y^{x+\varepsilon h} - Y^x)\cdot\alpha_{\mathcal{Y}}\big)(s) + A_{s,\mathcal{Z}}\big((Z^{x+\varepsilon h} - Z^x)\cdot\alpha_{\mathcal{Z}}\big)(s)\Big]\mathrm{d}s \\
&\quad - \int_t^T \Big\langle(\nabla f)\big(s,\Theta^x(s)\big), \big((\nabla X^x h\cdot\alpha_{\mathcal{X}})(s), (Ph\cdot\alpha_{\mathcal{Y}})(s), (Qh\cdot\alpha_{\mathcal{Z}})(s)\big)\Big\rangle\mathrm{d}s \\
&\quad - \int_t^T V_s\mathrm{d}W_s \\
&= \xi + \int_t^T \Phi\Big(s, (\tilde{X}\cdot\alpha_{\mathcal{X}})(s), (U\cdot\alpha_{\mathcal{Y}})(s), (V\cdot\alpha_{\mathcal{Z}})(s)\Big)\mathrm{d}s - \int_t^T V_s\mathrm{d}W_s,
\end{aligned}
\tag{5.43}
$$

with $\tilde{X}$ given in (5.42), $\Phi(t,x,y,z) := R_t + xA_{t,\mathcal{X}} + yA_{t,\mathcal{Y}} + zA_{t,\mathcal{Z}}$ and

$$
\begin{aligned}
R_t := &-\Big\langle(\nabla f)\big(t,\Theta^x(t)\big), \big((\nabla X^x h\cdot\alpha_{\mathcal{X}})(t), (Ph\cdot\alpha_{\mathcal{Y}})(t), (Qh\cdot\alpha_{\mathcal{Z}})(t)\big)\Big\rangle \\
&+ A_{t,\mathcal{X}}(\nabla X^x\cdot\alpha_{\mathcal{X}})(t) + A_{t,\mathcal{Y}}(Ph\cdot\alpha_{\mathcal{Y}})(t) + A_{t,\mathcal{Z}}(Qh\cdot\alpha_{\mathcal{Z}})(t).
\end{aligned}
$$

Our aim now is to apply the results of Section 5.2 to the family (indexed by ε) of auxiliary delay BSDEs (5.43). In view of the uniform boundedness of the processes A and the linearity of the driver Φ, we can repeat the arguments used in the proof of Corollary 5.3.1 to conclude that under the assumptions here, the data of BSDE (5.43) (i.e. Lipschitz constant, delay measure and terminal time) satisfy uniformly in ε the assumptions of Corollary 5.3.1 as well.

Applying the a priori estimate of Proposition 5.2.2 or the moment estimate from Corollary 5.2.1 to the BSDE (5.43) and taking into account that Φ satisfies (F2), we get

$$
\begin{aligned}
\|U\|^p_{\mathcal{S}^p_\beta} + \|V\|^p_{\mathcal{H}^p_\beta} &\leq C_p\Big\{\mathbb{E}\Big[(e^{\beta T}|\xi|^2)^{p/2}\Big] + \mathbb{E}\Big[\Big(\int_0^T e^{\beta s}\big|\Phi\big(s,(\tilde{X}\cdot\alpha_{\mathcal{X}})(s),0,0\big)\big|\mathrm{d}s\Big)^p\Big]\Big\} \\
&\leq C\Big\{\mathbb{E}\Big[(e^{\beta T}|\xi|^2)^{p/2}\Big] + \|\tilde{X}\|^2_{\mathcal{H}^p_\beta} + \mathbb{E}\Big[\Big(\int_0^T e^{\beta s}|R_s|\mathrm{d}s\Big)^p\Big]\Big\},
\end{aligned}
\tag{5.44}
$$

for some constant $C > 0$ (where we have used that $A_{\cdot,\mathcal{X}}$ is uniformly bounded). We proceed to compute the limit of each term on the right-hand side of (5.44) as ε tends to zero.

Let us deal with the second term of the right-hand side of (5.44). Define

$$
\hat{\sigma}_t := \int_0^1 \nabla\sigma\big(t, X^x_t + \theta(X^{x+\varepsilon h}_t - X^x_t)\big)\mathrm{d}\theta \quad\text{and}\quad \hat{b}_t := \int_0^1 \nabla b\big(t, X^x_t + \theta(X^{x+\varepsilon h}_t - X^x_t)\big)\mathrm{d}\theta.
$$

Note that $\tilde{X}\in\mathcal{S}^p$ for any $p\geq 2$ (see (5.42)) and solves the linear SDE

$$
\tilde{X}_t = J_t + \int_0^t \hat{\sigma}_s\tilde{X}_s\mathrm{d}W_s + \int_0^t \hat{b}_s\tilde{X}_s\mathrm{d}s,
\tag{5.45}
$$

where J is given by

$$J_t := \int_0^t [\, \nabla X_s^x h(\hat{\sigma}_s - \nabla\sigma(s, X_s^x))\,]\mathrm{d}W_s + \int_0^t [\, \nabla X_s^x h\big(\hat{b}_s - \nabla b(s, X_s^x)\big)\,]\mathrm{d}s.$$

Given the properties of ∇X and the fact that $\hat{b}, \hat{\sigma}, \nabla b$, and $\nabla\sigma$ are uniformly bounded, we have that $J \in \mathcal{S}_0^p$ for any $p \geq 2$. Indeed, Doob's inequality leads to

$$\mathbb{E}\Big[\Big(\sup_{t\in[0,T]} \Big|\int_0^t [\, \nabla X_s^x h\big(\hat{\sigma}_s - \nabla\sigma(s, X_s^x)\big)\,]\mathrm{d}W_s\Big|^2\Big)^{p/2}\Big] \leq C\big\|\nabla X^x h\big(\hat{\sigma} - \nabla\sigma(\cdot, X^x)\big)\big\|_{\mathcal{H}^p}^p < \infty.$$

Moreover, Lebesgue's dominated convergence theorem yields

$$\lim_{\varepsilon\to 0} \|\nabla X^x h\big(\hat{\sigma} - \nabla\sigma(\cdot, X^x)\big)\|_{\mathcal{H}^p}^p = 0.$$

Similarly, making use of Jensen's inequality, we see that the finite variation part of J is an element of $\mathcal{S}_0^p(\mathbb{R})$ and

$$\lim_{\varepsilon\to 0} \|J\|_{\mathcal{S}_0^p} = 0.$$

Now we derive the following estimate for $\tilde{X}$ in terms of the norm of J

$$\|\tilde{X}\|_{\mathcal{S}_\beta^p} \leq C\,\mathbb{E}[\sup_{t\in[0,T]} |\tilde{X}_t|^p] \leq C\,\|J\|_{\mathcal{S}_0^p}, \tag{5.46}$$

which will show that $\lim_{\varepsilon\to 0}\|\tilde{X}\|_{\mathcal{S}_\beta^p} = 0$. Indeed, (5.45) implies that

$$\mathbb{E}[\sup_{0\leq r\leq t} |\tilde{X}_r|^p] \leq C\,\mathbb{E}\Big[\sup_{0\leq r\leq t} |J_r|^p + \sup_{0\leq r\leq t}\Big|\int_0^r [\,\hat{\sigma}_s\tilde{X}_s\,]\mathrm{d}W_s\Big|^p + \sup_{0\leq r\leq t}\Big|\int_0^r [\hat{b}_s\tilde{X}_s\,]\mathrm{d}s\Big|^p\Big].$$

Applying the BDG inequality to the second term on the right-hand side, we get

$$\mathbb{E}[\sup_{0\leq r\leq t} |\tilde{X}_r|^p] \leq C\,\mathbb{E}\Big[\sup_{0\leq r\leq t} |J_r|^p + \Big|\int_0^t |\hat{\sigma}_s\tilde{X}_s|^2\mathrm{d}s\Big|^{p/2} + \sup_{0\leq r\leq t}\Big|\int_0^r [\hat{b}_s\tilde{X}_s\,]\mathrm{d}s\Big|^p\Big].$$

Jensen's inequality and the fact that $\hat{\sigma}$ and $\hat{b}$ are bounded imply

$$\mathbb{E}\Big[\sup_{0\leq r\leq t} |\tilde{X}_r|^p\Big] \leq C\,\mathbb{E}\Big[\sup_{0\leq r\leq t} |J_r|^p + \int_0^t |\tilde{X}_s|^p\mathrm{d}s\Big],$$

hence

$$\mathbb{E}\Big[\sup_{0\leq r\leq t} |\tilde{X}_r|^p\Big] \leq C\,\Big\{\mathbb{E}[\sup_{0\leq r\leq t} |J_r|^p] + \int_0^t \mathbb{E}[\sup_{0\leq r\leq s} |\tilde{X}_r|^p]\mathrm{d}s\Big\}.$$

Now an application of Gronwall's lemma finally yields estimate (5.46) and thus we have $\lim_{\varepsilon\to 0}\|\tilde{X}\|_{\mathcal{S}_\beta^p} = 0$.

Let us consider the terminal condition term in (5.44). For

$$\hat{g} := \int_0^1 \nabla g\big(X_T^x + \theta(X_T^{x+\varepsilon h} - X_T^x)\big)\mathrm{d}\theta,$$

we have

$$\begin{aligned}
\mathbb{E}\Big[(e^{\beta T}|\xi|^2)^{p/2}\Big] &= e^{\beta T p/2}\Big\|\hat{g}\Big(\frac{X_T^{x+\varepsilon h} - X_T^x}{\varepsilon} - \nabla X_T^x h\Big) + \big(\hat{g} - \nabla g(X_T^x)\big)\nabla X_T^x h\Big\|_{L^p}^p \\
&\le C\Big\{\Big\|\frac{X_T^{x+\varepsilon h} - X_T^x}{\varepsilon} - \nabla X_T^x h\Big\|_{L^p}^p + \big\|\,|\nabla X_T^x h|\,|\hat{g} - \nabla g(X_T^x)|\,\big\|_{L^p}^p \\
&\le C\Big\{\|\tilde{X}\|_{\mathcal{S}_0^p}^p + \big\|\,|\nabla X_T^x h|\,|\hat{g} - \nabla g(X_T^x)|\,\big\|_{L^p}^p\Big\}\Big\} \longrightarrow 0 \text{ as } \varepsilon \to 0,
\end{aligned}$$

where we have used Lebesgue's dominated convergence theorem for the second summand and the estimate obtained above on the norm of $\tilde{X}$ for the first one.

Now, we deal with the last term on the right-hand side of (5.44). To this end, we have

$$\begin{aligned}
\mathbb{E}\Big[\Big(\int_0^T e^{\beta s}|R_s|\mathrm{d}s\Big)^p\Big] &\le C\,\mathbb{E}\Big[\Big(\int_0^T e^{\beta s}\big|\big(A_{s,\mathcal{X}} - \nabla_x f\big(s,\Theta^x(s)\big)\big)\,(\nabla X^x h\cdot\alpha_{\mathcal{X}})(s)\big|\,\mathrm{d}s\Big)^p\Big] \\
&\quad + C\,\mathbb{E}\Big[\Big(\int_0^T e^{\beta s}\big|\big(A_{s,\mathcal{Y}} - \nabla_y f\big(s,\Theta^x(s)\big)\big)\,(Ph\cdot\alpha_{\mathcal{Y}})(s)\big|\,\mathrm{d}s\Big)^p\Big] \\
&\quad + C\,\mathbb{E}\Big[\Big(\int_0^T e^{\beta s}\big|\big(A_{s,\mathcal{Z}} - \nabla_z f\big(s,\Theta^x(s)\big)\big)\,(Qh\cdot\alpha_{\mathcal{Z}})(s)\big|\,\mathrm{d}s\Big)^p\Big].
\end{aligned}$$

Standard arguments yield (recall that $\varepsilon > 0$ is implicitly contained in $A_{t,\mathcal{X}}$, see (5.41))

$$A_{t,\mathcal{X}} \longrightarrow \nabla_x f\big(t,\Theta^x(t)\big) \quad \text{as } \varepsilon \to 0 \text{ in probability, for a.a. } t \in [0,T].$$

Moreover, Proposition 5.2.2 and the previous calculations show that

$$\begin{aligned}
&\|Y^{x+\varepsilon h} - Y^x\|_{\mathcal{S}_\beta^p}^p + \|Z^{x+\varepsilon h} - Z^x\|_{\mathcal{H}_\beta^p}^p \\
&\quad \le C\left\{e^{\beta T p}\|g(X^{x+\varepsilon h}) - g(X^x)\|_{L^p}^p + \|X^{x+\varepsilon h} - X^x\|_{\mathcal{H}_\beta^p}^p\right\} \underset{\varepsilon\to 0}{\longrightarrow} 0,
\end{aligned}$$

for some positive constant C. This implies for a.a. $t \in [0,T]$

$$Y_t^{x+\varepsilon h} \to Y_t^x, \quad Z_t^{x+\varepsilon h} \to Z_t^x, \quad \text{as } \varepsilon \to 0 \text{ in probability.}$$

Since $\nabla_y f$, $\nabla_z f$ are continuous, it follows that for a.a. $t \in [0,T]$

$$\begin{aligned}
A_{t,\mathcal{Y}} &\longrightarrow \nabla_y f\big(t,\Theta^x(t)\big), \quad \text{as } \varepsilon \to 0 \text{ in probability,} \\
A_{t,\mathcal{Z}} &\longrightarrow \nabla_z f\big(t,\Theta^x(t)\big), \quad \text{as } \varepsilon \to 0 \text{ in probability.}
\end{aligned}$$

Thus, using Lemma 5.1.1 and the fact that P and Q are square integrable, Lebesgue's dominated convergence theorem (which also holds, if almost sure convergence is replaced by convergence in probability, cf. Shiryaev [121], remark on p. 258) yields

$\lim_{\varepsilon\to 0}\mathbb{E}\big[\big(\int_0^T e^{\beta s}|R_s|\mathrm{d}s\big)^p\big] = 0$. Now (5.44) yields

$$\lim_{\varepsilon\to 0}\big\{\|U\|^p_{\mathcal{S}^p_\beta} + \|V\|^p_{\mathcal{H}^p_\beta}\big\} = 0,$$

which finishes the proof. □

5.3.2 Strong differentiability

All previous assumptions on existence and uniqueness remain in force. In this section, our focus is on the smoothness properties of the paths associated to the processes (Y^x, Z^x). We assume throughout this section that $m = 1$, i.e. the value processes of the delay BSDE are one-dimensional. We start with the following result.

Proposition 5.3.2. *If $m = 1$ and if the assumptions of Corollary 5.3.1 are in force, we have for $x, x' \in \mathbb{R}^d$*

$$\mathbb{E}\big[\sup_{0\le t\le T}|X^x_t - X^{x'}_t|^q\big] \le C|x - x'|^q, \quad \textit{for any} \quad q \ge 2,$$

and for every $p > 2$

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big(e^{\beta t}|Y^x_t - Y^{x'}_t|^2\big)^{p/2}\Big] + \mathbb{E}\Big[\Big(\int_0^T e^{\beta s}|Z^x_s - Z^{x'}_s|^2\mathrm{d}s\Big)^{p/2}\Big] \le C|x - x'|^p.$$

Thus, for every $x \in \mathbb{R}^d$,

- *the mapping $x \mapsto Y^x$ from $\mathbb{R}^d$ to the space of càdlàg functions equipped with the topology given by the uniform convergence on compacts sets is continuous $\mathbb{P}$-almost surely,*
- *the mapping $x \mapsto Z^x$ is continuous from $\mathbb{R}^d$ to $L^2([0,T])$ $\mathbb{P}$-almost surely.*

In particular, for every $x \in \mathbb{R}^d$,

- *the mapping $x \mapsto Y^x_t$ from $\mathbb{R}^d$ to $\mathbb{R}$ is continuous for all $t \in [0,T]$, $\mathbb{P}$-almost surely,*
- *the mapping $x \mapsto Z^x_t(\omega)$ is continuous for every $x \in \mathbb{R}^d$ and $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$-almost all (t,ω).*

Proof. The estimate on the forward process is classical (see e.g. Protter [114, Theorem V.37 Equation (***) p. 309]). In this proof, $C > 0$ denotes a generic constant which may vary from line to line. We apply the a priori estimate from Proposition 5.2.2 and get

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le T}\big(e^{\beta t}|Y_t^x-Y_t^{x'}|^2\big)^{p/2}\Big]+\mathbb{E}\Big[\Big(\int_0^T e^{\beta s}|Z_s^x-Z_s^{x'}|^2\mathrm{d}s\Big)^{p/2}\Big]\\
&\quad\le C_p\Big\{\mathbb{E}\Big[\big(e^{\beta T}|g(X_T^x)-g(X_T^{x'})|^2\big)^{p/2}\Big]\\
&\qquad+\mathbb{E}\Big[\Big(\int_0^T e^{\frac{\beta}{2}s}\big|f\big(s,(X^x\cdot\alpha_{\mathcal{X}})(s),\zeta(s)\big)-f\big(s,(X^{x'}\cdot\alpha_{\mathcal{X}})(s),\zeta(s)\big)\big|\mathrm{d}s\Big)^p\Big]\Big\}\\
&\quad\le C\Big\{\mathbb{E}\Big[\big(e^{\beta T}|g(X_T^x)-g(X_T^{x'})|^2\big)^{p/2}\Big]\\
&\qquad+\mathbb{E}\Big[\Big(\int_0^T e^{\beta s}|f(s,(X^x\cdot\alpha_{\mathcal{X}})(s),\zeta(s))-f(s,(X^{x'}\cdot\alpha_{\mathcal{X}})(s),\zeta(s))|^2\mathrm{d}s\Big)^{p/2}\Big]\Big\},
\end{aligned}$$

with $\zeta(\cdot):=\big((Y^{x'}\cdot\alpha_{\mathcal{Y}})(\cdot),(Z^{x'}\cdot\alpha_{\mathcal{Z}})(\cdot)\big)$. Using the mean value theorem and the boundedness of ∇f and ∇g (implied by the Lipschitz continuity of f and g), we deduce

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le T}\big(e^{\beta t}|Y_t^x-Y_t^{x'}|^2\big)^{p/2}\Big]+\mathbb{E}\Big[\Big(\int_0^T e^{\beta s}|Z_s^x-Z_s^{x'}|^2\mathrm{d}s\Big)^{p/2}\Big]\\
&\quad\le C\Big\{\mathbb{E}\Big[\big(e^{\beta T}|X_T^x-X_T^{x'}|^2\big)^{p/2}\Big]+\mathbb{E}\Big[\Big(\int_0^T e^{\beta s}|((X^x-X^{x'})\cdot\alpha_{\mathcal{X}})(s)|^2\mathrm{d}s\Big)^{p/2}\Big]\Big\}\\
&\quad\le C\Big\{\mathbb{E}\Big[\big(e^{\beta T}|X_T^x-X_T^{x'}|^2\big)^{p/2}\Big]+\mathbb{E}\Big[\Big(\int_0^T e^{\beta s}|X_s^x-X_s^{x'}|^2\mathrm{d}s\Big)^{p/2}\Big]\Big\}\\
&\quad\le C|x-x'|^p,
\end{aligned}$$

where the last two lines follow by applying the change of integration from (5.10) and the first claim of the proposition. The continuity properties of the mappings $x\mapsto Y^x$ and $x\mapsto Z^x$ are now obtained by an application of Kolmogorov's continuity criterion (see for example Protter [114, IV.7 Corollary 1]). □

If the generator exhibits additional regularity, it even turns out that the paths of $x\mapsto Y^x$ are continuously differentiable.

Theorem 5.3.1. *Let $\beta>0$ and assume that the conditions of Proposition 5.3.1 can be verified for some $\widehat{p}>4$. Assume moreover that all (spatial) second order partial derivatives of b,σ,g and f exist, are continuous and uniformly bounded. Then, for any $(x,\varepsilon),(x',\varepsilon')\in\mathbb{R}^d\times(0,\infty)$, $h\in\mathbb{R}^d$ and $p\in(2,\widehat{p}/2]$ it holds that*

$$\mathbb{E}\Big[\sup_{0\le t\le T}\Big(e^{\beta t}\Big|\frac{Y_t^{x+\varepsilon h}-Y_t^x}{\varepsilon}-\frac{Y_t^{x'+\varepsilon' h}-Y_t^{x'}}{\varepsilon'}\Big|^2\Big)^{p/2}\Big]\le C\left(|x-x'|^2+|\varepsilon-\varepsilon'|^2\right)^{p/2}.$$

Thus, $\nabla_x Y^x$ is in $\mathcal{H}_\beta^{\widehat{p}}$ and the mapping $x\mapsto Y_t^x(\omega)$ is continuously differentiable for all $t\in[0,T]$, $\mathbb{P}$-almost surely.

Proof of Theorem 5.3.1. Let $p>2$, $t\in[0,T]$ and $h\in\mathbb{R}^d\setminus\{0\}$. Let $C>0$ denote a generic constant which can vary from line to line. For $(x,\varepsilon)\in\mathbb{R}^d\times(0,\infty)$ let $U^{x,\varepsilon}:=\frac{Y^{x+\varepsilon h}-Y^x}{\varepsilon}$, $V^{x,\varepsilon}:=\frac{Z^{x+\varepsilon h}-Z^x}{\varepsilon}$, $\xi^{x,\varepsilon}:=\frac{g(X_T^{x+\varepsilon h})-g(X_T^x)}{\varepsilon}$ and $\tilde{X}^{x,\varepsilon}:=\frac{X^{x+\varepsilon h}-X^x}{\varepsilon}$. Using the notation from the proof of Proposition 5.3.1, the pair $(U^{x,\varepsilon}, V^{x,\varepsilon})$ satisfies the BSDE

$$U_t^{x,\varepsilon} = \xi^{x,\varepsilon} + \int_t^T \Phi(s, \zeta^{x,\varepsilon}(s))\mathrm{d}s - \int_t^T V_s^{x,\varepsilon}\mathrm{d}W_s,$$

with $\zeta^{x,\varepsilon}(t) := \big((U^{x,\varepsilon}\cdot\alpha_{\mathcal{Y}})(t), (V^{x,\varepsilon}\cdot\alpha_{\mathcal{Z}})(t)\big)$ and $\Phi(t,y,z) := (\tilde{X}^{x,\varepsilon}\cdot\alpha_{\mathcal{X}})(t)A_{t,\mathcal{X}}^{x,\varepsilon} + yA_{t,\mathcal{Y}}^{x,\varepsilon} + zA_{t,\mathcal{Z}}^{x,\varepsilon}$. Note that the terms $A_{\cdot,*}^{x,\varepsilon}$ with $* = \mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are given by (5.41). For every choice of (x,ε) we emphasize that the arguments used in the proof of Corollary 5.3.1 and Proposition 5.3.1 hold true for the above auxiliary BSDE in all matters concerning the applicability of the a priori estimate of Proposition 5.2.2.

Let another pair $(x',\varepsilon') \in \mathbb{R}^d \times (0,\infty)$ be given. Applying Proposition 5.2.2 yields

$$\|U^{x,\varepsilon} - U^{x',\varepsilon'}\|_{\mathcal{S}_\beta^p}^p \le C_p\Big\{\mathbb{E}\Big[\big(e^{\beta T}|\xi^{x,\varepsilon} - \xi^{x',\varepsilon'}|^2\big)^{p/2}\Big] + \mathbb{E}\Big[\Big(\int_0^T e^{\frac{\beta}{2}s}|\delta_2\Phi(s)|\mathrm{d}s\Big)^p\Big]\Big\},$$

with

$$\begin{aligned}\delta_2\Phi(t) := {}& (\tilde{X}^{x,\varepsilon}\cdot\alpha_{\mathcal{X}})(t)A_{t,\mathcal{X}}^{x,\varepsilon} - (\tilde{X}^{x',\varepsilon'}\cdot\alpha_{\mathcal{X}})(t)A_{t,\mathcal{X}}^{x',\varepsilon'} \\ &+ (U^{x',\varepsilon'}\cdot\alpha_{\mathcal{Y}})(t)(A_{t,\mathcal{Y}}^{x,\varepsilon} - A_{t,\mathcal{Y}}^{x',\varepsilon'}) + (V^{x',\varepsilon'}\cdot\alpha_{\mathcal{Z}})(t)(A_{t,\mathcal{Z}}^{x,\varepsilon} - A_{t,\mathcal{Z}}^{x',\varepsilon'}).\end{aligned}$$

Using the hypotheses on f (i.e. all partial derivatives up to order two are bounded), we find

$$\begin{aligned}|\delta_2\Phi(t)| \le C\Big\{&|((\tilde{X}^{x,\varepsilon} - \tilde{X}^{x',\varepsilon'})\cdot\alpha_{\mathcal{X}})(t)||A_{t,\mathcal{X}}^{x,\varepsilon}| + |(\tilde{X}^{x',\varepsilon'}\cdot\alpha_{\mathcal{X}})(t)||A_{t,\mathcal{X}}^{x,\varepsilon} - A_{t,\mathcal{X}}^{x',\varepsilon'}| \\ &+ |(U^{x',\varepsilon'}\cdot\alpha_{\mathcal{Y}})(s)||A_{t,\mathcal{Y}}^{x,\varepsilon} - A_{t,\mathcal{Y}}^{x',\varepsilon'}| + |(V^{x',\varepsilon'}\cdot\alpha_{\mathcal{Z}})(t)||A_{t,\mathcal{Z}}^{x,\varepsilon} - A_{t,\mathcal{Z}}^{x',\varepsilon'}|\Big\}.\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}&\|U^{x,\varepsilon} - U^{x',\varepsilon'}\|_{\mathcal{S}_\beta^p}^p \\ &\le C\Big\{\|\xi^{x,\varepsilon} - \xi^{x',\varepsilon'}\|_{L^p}^p + \mathbb{E}\Big[\Big(\int_0^T e^{\frac{\beta}{2}s}|((\tilde{X}^{x,\varepsilon} - \tilde{X}^{x',\varepsilon'})\cdot\alpha_{\mathcal{X}})(s)||A_{s,\mathcal{X}}^{x,\varepsilon}|\mathrm{d}s\Big)^p\Big] \\ &\quad + \mathbb{E}\Big[\Big(\int_0^T e^{\frac{\beta}{2}s}|(\tilde{X}^{x',\varepsilon'}\cdot\alpha_{\mathcal{X}})(s)||A_{s,\mathcal{X}}^{x,\varepsilon} - A_{s,\mathcal{X}}^{x',\varepsilon'}|\mathrm{d}s\Big)^p\Big] \\ &\quad + \mathbb{E}\Big[\Big(\int_0^T e^{\frac{\beta}{2}s}|(U^{x',\varepsilon'}\cdot\alpha_{\mathcal{Y}})(s)||A_{s,\mathcal{Y}}^{x,\varepsilon} - A_{s,\mathcal{Y}}^{x',\varepsilon'}|\mathrm{d}s\Big)^p\Big] \\ &\quad + \mathbb{E}\Big[\Big(\int_0^T e^{\frac{\beta}{2}s}|(V^{x',\varepsilon'}\cdot\alpha_{\mathcal{Z}})(s)||A_{s,\mathcal{Z}}^{x,\varepsilon} - A_{s,\mathcal{Z}}^{x',\varepsilon'}|\mathrm{d}s\Big)^p\Big]\Big\} \\ &\le C\Big\{\|\xi^{x,\varepsilon} - \xi^{x',\varepsilon'}\|_{L^p}^p + \|\tilde{X}^{x,\varepsilon} - \tilde{X}^{x',\varepsilon'}\|_{\mathcal{H}_\beta^{2p}}^p\|A_{\cdot,\mathcal{X}}^{x,\varepsilon}\|_{\mathcal{H}_\beta^{2p}}^p + \|\tilde{X}^{x',\varepsilon'}\|_{\mathcal{H}_\beta^{2p}}^p\|A_{\cdot,\mathcal{X}}^{x,\varepsilon} - A_{\cdot,\mathcal{X}}^{x',\varepsilon'}\|_{\mathcal{H}_\beta^{2p}}^p \\ &\qquad + \|U^{x',\varepsilon'}\|_{\mathcal{H}_\beta^{2p}}^p\|A_{\cdot,\mathcal{Y}}^{x,\varepsilon} - A_{\cdot,\mathcal{Y}}^{x',\varepsilon'}\|_{\mathcal{H}_\beta^{2p}}^p + \|V^{x',\varepsilon'}\|_{\mathcal{H}_\beta^{2p}}^p\|A_{\cdot,\mathcal{Z}}^{x,\varepsilon} - A_{\cdot,\mathcal{Z}}^{x',\varepsilon'}\|_{\mathcal{H}_\beta^{2p}}^p\Big\},\end{aligned}$$

where for each term we used twice the Cauchy-Schwarz inequality, the fact that $e^{\frac{\beta}{2}t} \le e^{\beta t}$ and the inequality from (5.10). Since $(U^{x',\varepsilon'}, V^{x',\varepsilon'})$ is a solution in $\mathcal{S}_\beta^p \times \mathcal{H}_\beta^p$ of a

BSDE, it follows from Corollary 5.2.1 that the quantities $\mathbb{E}\big[\big(\int_0^T e^{\beta s}|U_s^{x',\varepsilon'}|^2 \mathrm{d}s\big)^p\big]$ and $\mathbb{E}\big[\big(\int_0^T e^{\beta s}|V_s^{x',\varepsilon'}|^2 \mathrm{d}s\big)^p\big]$ are finite and uniformly bounded in ε'. By the assumptions on b and σ, we have

$$\mathbb{E}\Big[\Big(\int_0^T e^{\beta s}|\tilde{X}_s^{x',\varepsilon'}|^2 \mathrm{d}s\Big)^p\Big]^{1/2} < \infty.$$

In addition, by the boundedness of ∇f we have for $* = \mathcal{X}, \mathcal{Y}, \mathcal{Z}$ that $|A^{x,\varepsilon}_{\cdot,*}|$ and $|A^{x',\varepsilon'}_{\cdot,*}|$ are uniformly bounded with respect to x and ε. Thus, the estimate reduces to

$$\begin{aligned}\|U^{x,\varepsilon} - U^{x',\varepsilon'}\|^p_{\mathcal{S}^p_\beta} \le C\Big\{&\|\xi^{x,\varepsilon} - \xi^{x',\varepsilon'}\|^p_{L^p} + \|\tilde{X}^{x,\varepsilon} - \tilde{X}^{x',\varepsilon'}\|^p_{\mathcal{H}^{2p}_\beta} + \|A^{x,\varepsilon}_{\cdot,\mathcal{X}} - A^{x',\varepsilon'}_{\cdot,\mathcal{X}}\|^p_{\mathcal{H}^{2p}_\beta} \\ &+ \|A^{x,\varepsilon}_{\cdot,\mathcal{Y}} - A^{x',\varepsilon'}_{\cdot,\mathcal{Y}}\|^p_{\mathcal{H}^{2p}_\beta} + \|A^{x,\varepsilon}_{\cdot,\mathcal{Z}} - A^{x',\varepsilon'}_{\cdot,\mathcal{Z}}\|^p_{\mathcal{H}^{2p}_\beta}\Big\}.\end{aligned} \tag{5.47}$$

By the mean value theorem and the fact that the second order partial derivatives are bounded it follows that

$$\begin{aligned}&|A^{x,\varepsilon}_{t,\mathcal{X}} - A^{x',\varepsilon'}_{t,\mathcal{X}}| + |A^{x,\varepsilon}_{t,\mathcal{Y}} - A^{x',\varepsilon'}_{t,\mathcal{Y}}| + |A^{x,\varepsilon}_{t,\mathcal{Z}} - A^{x',\varepsilon'}_{t,\mathcal{Z}}| \\ &\quad\le C\Big\{\big(|X^{x+\varepsilon h} - X^{x'+\varepsilon' h}|\cdot\alpha_{\mathcal{X}}\big)(t) + \big(|Y^{x+\varepsilon h} - Y^{x'+\varepsilon' h}|\cdot\alpha_{\mathcal{Y}}\big)(t) \\ &\qquad + \big(|Z^{x+\varepsilon h} - Z^{x'+\varepsilon' h}|\cdot\alpha_{\mathcal{Z}}\big)(t) + \big(|X^{x} - X^{x'}|\cdot\alpha_{\mathcal{X}}\big)(t) \\ &\qquad + \big(|Y^{x} - Y^{x'}|\cdot\alpha_{\mathcal{Y}}\big)(t) + \big(|Z^{x} - Z^{x'}|\cdot\alpha_{\mathcal{Z}}\big)(t)\Big\}.\end{aligned}$$

Plugging the right-hand side of this inequality into (5.47) and using Lemma 5.1.1, we get

$$\begin{aligned}\|U^{x,\varepsilon} - U^{x',\varepsilon'}\|^p_{\mathcal{S}^p_\beta} \le C\Big\{&\|\xi^{x,\varepsilon} - \xi^{x',\varepsilon'}\|^p_{L^p} + \|\tilde{X}^{x,\varepsilon} - \tilde{X}^{x',\varepsilon'}\|^p_{\mathcal{H}^{2p}_\beta} + \|X^{x} - X^{x'}\|^p_{\mathcal{H}^{2p}_\beta} \\ &+ \|X^{x+\varepsilon h} - X^{x'+\varepsilon' h}\|^p_{\mathcal{H}^{2p}_\beta} + \|Y^{x+\varepsilon h} - Y^{x'+\varepsilon' h}\|^p_{\mathcal{H}^{2p}_\beta} \\ &+ \|Z^{x+\varepsilon h} - Z^{x'+\varepsilon' h}\|^p_{\mathcal{H}^{2p}_\beta} + \|Y^{x} - Y^{x'}\|^p_{\mathcal{H}^{2p}_\beta} + \|Z^{x} - Z^{x'}\|^p_{\mathcal{H}^{2p}_\beta}\Big\}.\end{aligned}$$

Since b, σ and g are twice continuously differentiable with bounded derivatives we have the estimate

$$\mathbb{E}\big[\,|\xi^{x,\varepsilon} - \xi^{x',\varepsilon'}|^p\big] \le C(|x-x'|^2 + |\varepsilon - \varepsilon'|^2)^{p/2},$$

which can be found in Lemma 7.4 in Ankirchner et al. [4]. This result combined with Proposition 5.3.2 leads to

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big(e^{\beta t}|U_t^{x,\varepsilon} - U_t^{x',\varepsilon'}|^2\big)^{p/2}\Big] \le C\big(|x-x'|^2 + |\varepsilon-\varepsilon'|^2\big)^{p/2}.$$

The last claim of the theorem follows from Kolmogorov's continuity criterion (see for example Protter [114, IV.7 Corollary 1]). □

5.4 Representation formulas and path regularity

One of the fundamental results in the setting of FBSDEs concerns the relationship between the Malliavin and the variational (also called classical) derivatives of the solution process: the Malliavin derivative of the solution of the BSDE can be expressed as a product of the BSDE's variational derivatives (with respect to the initial parameter of the SDE) and the variational derivatives of the forward diffusion. This relationship is known to hold both in the standard Lipschitz generator setting (see Proposition 5.9 in El Karoui et al. [50]) as well as the quadratic generator case (see e.g. Theorem 2.9 in Imkeller and Dos Reis [66]) for classical BSDE without time delayed generators.

In this section we show that this relationship still holds true for decoupled FBSDEs with time delayed generators. Such a result is somewhat surprising since it heavily depends on the Markov structure. Obviously, because the drivers of time delayed BSDEs depend on the past, the Markov property fails to materialize for time delayed BSDE. However, imperative for this relationship to hold is the fact that the forward process X is Markovian along with a good behavior of the terminal condition.

As in the previous section, whenever we consider the delay FBSDE (5.33)-(5.34), we assume that all conditions ensuring existence and uniqueness of a solution (X, Y, Z) are in force. Moreover, since for $\beta \geq 0$, all β-norms are equivalent, in the following we content ourselves with giving results for $\beta = 0$. Recall that we assume $m = 1$, i.e. the value process of the delay BSDE, Y, is *not* vector valued.

Malliavin differentiability of FBSDEs with time delayed generators

We recall Theorem 4.1 from Delong and Imkeller [43], modified to our the FBSDE setting. Theorem 4.1 from Delong and Imkeller [43] shows that the solutions of time delayed BSDEs are Malliavin differentiable, and as a consequence, it can be deduced that the solution of the time delayed FBSDE (5.33)-(5.34) is also Malliavin differentiable. Under the condition (F3) on the coefficients of the forward equation (5.33), the Malliavin differentiability of the forward process X is a standard result, see for instance Theorem 2.2.1 in Nualart [98]. We denote the solution to the equations (5.33)-(5.34) by (X, Y, Z). The next result states the Malliavin differentiability of (X, Y, Z). Let us define for $0 \leq u \leq t \leq T$

$$\begin{aligned}(D_u\Theta)(t) &= \big((D_uX \cdot \alpha_{\mathcal{X}})(t), (D_uY \cdot \alpha_{\mathcal{Y}})(t), (D_uZ \cdot \alpha_{\mathcal{Z}})(t)\big) \\ &= \Big(\int_{-T}^{0} D_uX_{t+v}\alpha_{\mathcal{X}}(\mathrm{d}v), \int_{-T}^{0} D_uY_{t+v}\alpha_{\mathcal{Y}}(\mathrm{d}v), \int_{-T}^{0} D_uZ_{t+v}\alpha_{\mathcal{Z}}(\mathrm{d}v)\Big). \qquad (5.48)\end{aligned}$$

We define [4] the space $\mathbb{L}_{1,2}$ as the space of progressively measurable processes $X \in \mathcal{H}^2$ that are Malliavin differentiable and equipped with the norm $\|X\|_{\mathbb{L}_{1,2}} = \mathbb{E}[\int_0^T |X_s|^2\mathrm{d}s + \int_0^T\int_0^T |D_uX_s|^2\mathrm{d}s\mathrm{d}u]^{1/2}$.

[4] See Section 2.2 of Imkeller and Dos Reis [66], Section 5.2 of El Karoui et al. [50] or Nualart [98].

Theorem 5.4.1. *Let $p = 2$, $m = 1$ and assume that the conditions of Corollary 5.3.1 hold true. Then (X, Y, Z) is Malliavin differentiable and the derivatives (DX, DY, DZ) solve uniquely in $\mathbb{L}_{1,2} \times \mathbb{L}_{1,2} \times \mathbb{L}_{1,2}$ the time delayed FBSDE*

$$D_u X_t = \sigma(u, X_u) + \int_u^t \nabla_x b(s, X_s) D_u X_s \mathrm{d}s + \int_u^t \nabla_x \sigma(s, X_s) D_u X_s \mathrm{d}W_s, \tag{5.49}$$

$$D_u Y_t = \nabla g(X_T) D_u X_T - \int_t^T D_u Z_s \mathrm{d}W_s + \int_t^T \big\langle (\nabla f)\big(s, \Theta(s)\big), (D_u \Theta)(s) \big\rangle \mathrm{d}s, \tag{5.50}$$

for $0 \le u \le t \le T$ (zero otherwise) with Θ and $D\Theta$ given by (5.35) *and* (5.48) *respectively. Furthermore, $\{D_t Y_t : t \in [0, T]\}$ is a version of $\{Z_t : t \in [0, T]\}$.*

Proof. The results concerning the forward component are well known, see e.g. Nualart [98] or Imkeller and Dos Reis [66]. The conditions of Corollary 5.3.1 ensure that Theorem 4.1 from Delong and Imkeller [43] can be applied. Hence, Y and Z are Malliavin differentiable. Now the representation of Z as the trace of the Malliavin derivative of Y follows from standard results. □

The representation formulas

Let us now give the representation formulas for (5.49) and (5.50) which are expressed in terms of the variational equations $\nabla X, \nabla Y$ and ∇Z.

Theorem 5.4.2. *Let the conditions of Theorem 4.1 hold. Let (X, Y, Z), $(\nabla X, \nabla Y, \nabla Z)$ and (DX, DY, DZ) denote the solutions of FBSDE* (5.33)-(5.34), (5.36)-(5.37) *and* (5.49)-(5.50) *respectively. Then, the following representation formulas hold:*

$$D_u X_t = \nabla X_t (\nabla X_u)^{-1} \sigma(u, X_u) \mathbb{1}_{\{u \le t\}}, \qquad t, u \in [0, T],\ \mathrm{d}\mathbb{P} - a.s. \tag{5.51}$$

$$D_u Y_t = \nabla Y_t (\nabla X_u)^{-1} \sigma(u, X_u) \mathbb{1}_{\{u \le t\}}, \qquad t, u \in [0, T],\ \mathrm{d}\mathbb{P} - a.s.$$

$$Z_t = \nabla Y_t (\nabla X_t)^{-1} \sigma(t, X_t), \qquad t \in [0, T],\ \mathrm{d}\mathbb{P} \otimes \mathrm{d}t - a.s. \tag{5.52}$$

$$D_u Z_t = \nabla Z_t (\nabla X_u)^{-1} \sigma(t, X_u) \mathbb{1}_{\{u \le t\}}, \qquad t, u \in [0, T],\ \mathrm{d}\mathbb{P} \otimes \mathrm{d}t - a.s.$$

Proof. Similar as in Theorem 5.4.1 we remark that the properties of the forward component are well known and hence equality (5.51) holds, see e.g. Nualart [98] or Imkeller and Dos Reis [66]. Theorem 5.4.1 yields that (DX, DY, DZ) is the unique solution of the time delayed FBSDE (5.49)-(5.50). Let $t \in [0, T]$ and $u \in [0, t]$. We define the processes

$$U_{u,t} = \nabla Y_t (\nabla X_u)^{-1} \sigma(X_u) \mathbb{1}_{\{u \le t\}} \quad \text{and} \quad V_{u,t} = \nabla Z_t (\nabla X_u)^{-1} \sigma(X_u) \mathbb{1}_{\{u \le t\}},$$

and for $s \in [0, T]$, we set $D_u X(s) = \int_{-T}^0 D_u X_{s+v} \alpha_{\mathcal{X}}(\mathrm{d}v)$,

$$U_u(s) = \int_{-T}^0 U_{u,s+v} \alpha_{\mathcal{Y}}(\mathrm{d}v) = \int_{-T}^0 \nabla Y_{s+v} \big(\nabla X_u\big)^{-1} \sigma(u, X_u) \mathbb{1}_{\{u \le s+v\}} \alpha_{\mathcal{Y}}(\mathrm{d}v),$$

$$V_u(s) = \int_{-T}^0 V_{u,s+v} \alpha_{\mathcal{Z}}(\mathrm{d}v) = \int_{-T}^0 \nabla Z_{s+v} \big(\nabla X_u\big)^{-1} \sigma(u, X_u) \mathbb{1}_{\{u \le s+v\}} \alpha_{\mathcal{Z}}(\mathrm{d}v),$$

which is consistent with the notation in (5.1). Now multiplying the BSDE (5.37) with $(\nabla X_u)^{-1}\sigma(u, X_u)$ and then using (5.51), we obtain for every $0 \le u \le t \le T$ $\mathrm{d}\mathbb{P}$-a.s.

$$U_{u,t} = \nabla g(X_T) D_u X_T - \int_t^T V_{u,s} \mathrm{d}W_s \\ + \int_t^T \big\langle (\nabla f)\big(s, \Theta(s)\big), \big(D_u X(s), U_u(s), V_u(s)\big)\big\rangle \mathrm{d}s,$$

where Θ is given by $\Theta(\cdot) = \big((X \cdot \alpha_{\mathcal{X}})(\cdot), (Y \cdot \alpha_{\mathcal{Y}})(\cdot), (Z \cdot \alpha_{\mathcal{Z}})(\cdot)\big)$ (compare with (5.35) from Section 5.3). Now Theorem 5.4.1 states that the solution of BSDE (5.50) is unique, hence (U, V) must coincide with (DY, DZ). Another way to see this is to apply the a priori estimates from Proposition 5.2.2 to (5.50) and the above BSDE. Formula (5.52) follows easily from a combination of the representation formula for $D_u Y_t$ combined with $D_t Y_t = Z_t$, $\mathrm{d}\mathbb{P} \otimes \mathrm{d}t$-a.s. (see Theorem 5.4.1). $\square$

Implications of the representation formula

The representation formulas from the previous theorem allow for a deeper analysis of the control process Z concerning its path properties.

Theorem 5.4.3. *Let $p \ge 2$, assume that $|f(\cdot, 0, 0, 0)|$ is uniformly bounded and that the conditions of Corollary 5.3.1 hold. Then for $p \ge 2$, the mapping $t \mapsto Z_t$ is continuous* $\mathrm{d}\mathbb{P}$*-a.s. Moreover, if $p > 2$, then we also have*

$$\|Z\|_{\mathcal{S}_0^q} < \infty \quad \textit{for } q \in [2, \frac{p}{2}).$$

In particular, there exists $r > 2$ such that $\mathbb{E}\big[\,|Y_t - Y_s|^r\big] \le C|t-s|^{r/2}$ for every $s, t \in [0, T]$, and Y has continuous paths.

Proof. It is straightforward to show that $\big(\nabla Y_t (\nabla X_t)^{-1} \sigma(t, X_t)\big)_{t \in [0,T]}$ is continuous. By assumption, σ is a continuous function and it is well known that both processes $(\nabla X)^{-1}$ and X have continuous paths. Now, ∇Y is continuous because its dynamics is given as a sum of a stochastic integral of a predictable process with respect to a Brownian motion (and thus a continuous martingale) and a Lebesgue integral with well behaved integrand. If two processes are versions of each other and one is continuous then they are in fact modifications of each other and hence Z has continuous paths. Now since Z has continuous paths, the representation formula (5.52) does not only hold $\mathrm{d}\mathbb{P} \otimes \mathrm{d}t$-almost surely but holds for all $t \in [0, T]$ and $\mathbb{P}$-almost all $\omega \in \Omega$. By the fact that $\nabla Y \in \mathcal{S}_0^p$ for some $p > 2$ (see Corollary 5.3.1 and Proposition 5.3.1), $(\nabla X)^{-1}, \sigma(\cdot, X) \in \mathcal{S}_0^r$ for $r \ge 2$ and that for any $q \ge 2$

$$\big|\nabla Y_t (\nabla X_t)^{-1} \sigma(t, X_t)\big|^q \le C\Big[\big|\nabla Y_t\big|^{2q} + \big|(\nabla X_t)^{-1}\big|^{4q} + \big|\sigma(t, X_t)\big|^{4q}\Big]$$

we conclude that $Z \in \mathcal{S}_0^q$ for every $q \in [2, \frac{p}{2})$.

The property concerning the increments of Y is straightforward to prove since we have $X, Y, Z \in \mathcal{S}_0^r$ for some $r > 2$. For $0 \le s \le t \le T$, we have (recall that $|f(\cdot,\Theta(\cdot))| \le |f(\cdot,\Theta(\cdot)) - f(\cdot,0,0,0)| + |f(\cdot,0,0,0)|$ and that $|f(\cdot,0,0,0)|$ is uniformly bounded)

$$Y_t - Y_s = 0 + \int_s^t f\big(u,\Theta(u)\big)\mathrm{d}u - \int_s^t Z_u \mathrm{d}W_u,$$

so using the assumptions and the BDG inequality, we get for a generic constant C which may vary from line to line

$$\begin{aligned}\mathbb{E}\big[\,|Y_t - Y_s|^r\big] &\le C\,\mathbb{E}\Big[\Big|\int_s^t f\big(u,\Theta(u)\big)\mathrm{d}u\Big|^r + \Big|\int_s^t Z_u \mathrm{d}W_u\Big|^r\Big] \\ &\le C\,|t-s|^{r/2}\big(1 + \|X\|^r_{\mathcal{S}_0^r} + \|Y\|^r_{\mathcal{S}_0^r} + \|Z\|^r_{\mathcal{S}_0^r}\big) + \mathbb{E}\Big[\Big(\int_s^t |Z_u|^2\mathrm{d}u\Big)^{r/2}\Big] \\ &\le C\,|t-s|^{r/2},\end{aligned}$$

where the last line follows by choosing $r \in [2, \frac{p}{2})$. This in particular yields the applicability of Kolmogorov's continuity criterion to Y. $\square$

The L^2-regularity result

We finish this section with the L^2-regularity result for the control component Z of the solution of the time delayed FBSDE. Let π be a partition of the time interval $[0,T]$ with N points and mesh size $|\pi|$. We define a family of random variables by

$$\bar{Z}^\pi_{t_i} = \frac{1}{t_{i+1}-t_i}\mathbb{E}\Big[\int_{t_i}^{t_{i+1}} Z_s \mathrm{d}s \Big| \mathcal{F}_{t_i}\Big], \text{ for all partition points } t_i,\ 0 \le i \le N-1.$$

The least squares estimate of $\frac{1}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}} Z_s\mathrm{d}s$ among square integrable $\mathcal{F}_{t_i}$-measurable random variables is given by $\bar{Z}^\pi_{t_i}$, i.e.

$$\mathbb{E}\Big[\,\Big|\frac{1}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}} Z_s\mathrm{d}s - \bar{Z}^\pi_{t_i}\Big|^2\Big] = \inf_{V \in L^2(\mathcal{F}_{t_i})} \mathbb{E}\Big[\,\Big|\frac{1}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}} Z_s\mathrm{d}s - V\Big|^2\Big]. \tag{5.53}$$

We associate the process $(\bar{Z}^\pi_t)_{t\in[0,T]}$ to $\{\bar{Z}^\pi_{t_i}\}_{i=0,\cdots,N-1}$ by constant interpolation $\bar{Z}^\pi_t = \bar{Z}^\pi_{t_i}$ for $t \in [t_i, t_{i+1})$, $0 \le i \le N-1$. Similarly, for the set of random variables $\{Z_{t_i} : t_i \in \pi\}$, we associate the process $(Z^\pi_t)_{t\in[0,T]}$ via $Z^\pi_t = Z^\pi_{t_i}$ for $t \in [t_i, t_{i+1})$, $0 \le i \le N-1$. The definition of the conditional expectation implies that for every $i = 0, \ldots, N-1$, we have

$$\mathbb{E}[\,|Z^\pi_{t_i}|^2] - 2\,\mathbb{E}[\,Z^\pi_{t_i}\,\bar{Z}^\pi_{t_i}\,] \ge -\mathbb{E}[\,|\bar{Z}^\pi_{t_i}|^2],$$

from which it follows that

$$\|Z - \bar{Z}^\pi\|_{\mathcal{H}^2} \le \|Z - Z^\pi\|_{\mathcal{H}^2} \to 0, \quad \text{as} \quad |\pi| \to 0.$$

By Theorem 5.4.3 we are able to determine explicitly the rate of convergence of the above limit. The following result extends Theorem 5.6 from Imkeller and Dos Reis [66] to the setting of FBSDE with time delayed generators.

Theorem 5.4.4 (L^2-regularity)**.** *Assume that the conditions of Theorem 5.4.3 hold for some $p>2$ and assume further that σ is $\frac{1}{2}$-Hölder continuous function in its time variable. Then, we have*

$$\max_{0\le i\le N-1}\Big\{\sup_{t_i\le t\le t_{i+1}}\mathbb{E}\big[\,|Y_t-Y_{t_i}|^2\,\big]\Big\}+\sum_{i=0}^{N-1}\mathbb{E}\Big[\int_{t_i}^{t_{i+1}}|Z_s-\bar Z^\pi_{t_i}|^2\mathrm{d}s\Big]\le C|\pi|.$$

Proof. The result concerning the Y component follows immediately from Theorem 5.4.3. As for the result for Z, let us remark that since $\bar Z^\pi$ is the best $\mathcal{H}^2$-approximation of Z over π in the sense of (5.53), it follows that

$$\sum_{i=0}^{N-1}\mathbb{E}\Big[\int_{t_i}^{t_{i+1}}|Z_s-\bar Z^\pi_{t_i}|^2\mathrm{d}s\Big]\le\sum_{i=0}^{N-1}\mathbb{E}\Big[\int_{t_i}^{t_{i+1}}|Z_s-Z_{t_i}|^2\mathrm{d}s\Big]=\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\mathbb{E}\big[\,|Z_s-Z_{t_i}|^2\big]\mathrm{d}s,$$

where the last equality follows from Fubini's theorem (recall that $Z\in\mathcal{S}^p_0$ for some $p>2$). Theorem 5.4.3 allows to use (5.52) to rewrite the difference inside the expectation. We have $Z_s-Z_{t_i}=I_1+I_2+I_3$ with $I_1=[\nabla Y_s-\nabla Y_{t_i}](\nabla X_{t_i})^{-1}\sigma(t_i,X_{t_i})$, $I_2=\nabla Y_s[(\nabla X_s)^{-1}-(\nabla X_{t_i})^{-1}]\sigma(t_i,X_{t_i})$, $I_3=\nabla Y_s(\nabla X_s)^{-1}[\sigma(s,X_s)-\sigma(t_i,X_{t_i})]$ and $s\in[t_i,t_{i+1}]$.

From the proof of part (ii) of Theorem 5.8 in Imkeller and Dos Reis [67], we get

$$\sum_{i=0}^{N-1}\mathbb{E}\Big[\int_{t_i}^{t_{i+1}}|I_2|^2\mathrm{d}s+\int_{t_i}^{t_{i+1}}|I_3|^2\mathrm{d}s\Big]\le C|\pi|.$$

The calculations that lead to the above result rely on known estimates for SDEs found for instance in Theorem 2.3 and 2.4 of Imkeller and Dos Reis [66] combined with the fact that $\nabla Y\in\mathcal{S}^p$ for some $p>2$.

Moreover, observe that we have

$$\mathbb{E}\Big[\,|(\nabla Y_s-\nabla Y_{t_i})(\nabla X_{t_i})^{-1}\sigma(t_i,X_{t_i})|^2\Big]=\mathbb{E}\Big[\,\mathbb{E}\big[\,|\nabla Y_s-\nabla Y_{t_i}|^2\big|\mathcal{F}_{t_i}\big]|(\nabla X_{t_i})^{-1}\sigma(t_i,X_{t_i})|^2\Big].\tag{5.54}$$

Writing the BSDE for the difference $\nabla Y_s-\nabla Y_{t_i}$ for $s\in[t_i,t_{i+1}]$ we get for a generic constant $C>0$

$$\begin{aligned}\mathbb{E}\Big[\,|\nabla Y_s-\nabla Y_{t_i}|^2\Big|\mathcal{F}_{t_i}\Big]&\le C\,\mathbb{E}\Big[\,\Big|\int_{t_i}^{s}\big\langle(\nabla f)\big(r,\Theta(r)\big),(\nabla\Theta)(r)\big\rangle\mathrm{d}r\Big|^2+\Big|\int_{t_i}^{s}\nabla Z_r\mathrm{d}W_r\Big|^2\Big|\mathcal{F}_{t_i}\Big]\\&\le C\,\mathbb{E}\Big[\,|\pi|\int_{t_i}^{t_{i+1}}\big|(\nabla\Theta)(r)\big|^2\mathrm{d}r+\int_{t_i}^{t_{i+1}}|\nabla Z_r|^2\mathrm{d}r\Big|\mathcal{F}_{t_i}\Big],\end{aligned}$$

which follows by applying the uniform boundedness of the derivatives of f, Jensen's inequality, Itô's isometry and maximizing over the time interval $[t_i,t_{i+1}]$. Combining this with (5.54) and the tower property of expectations, we obtain

$$
\begin{aligned}
&\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\Big[\, \mathbb{E}\Big[\, |\nabla Y_s - \nabla Y_{t_i}|^2 \Big| \mathcal{F}_{t_i}\Big] |(\nabla X_{t_i})^{-1}\sigma(t_i, X_{t_i})|^2\Big] \mathrm{d}s \\
&\quad \le C \sum_{i=0}^{N-1} |\pi| \mathbb{E}\Big[\Big(|\pi| \int_{t_i}^{t_{i+1}} \big|(\nabla\Theta)(r)|^2 \mathrm{d}r + \int_{t_i}^{t_{i+1}} |\nabla Z_r|^2 \mathrm{d}r\Big) |(\nabla X_{t_i})^{-1}\sigma(t_i, X_{t_i})|^2\Big] \\
&\quad \le |\pi| \mathbb{E}\Big[\sup_{0\le t\le T} |(\nabla X_t)^{-1}\sigma(t, X_t)|^2 \sum_{i=0}^{N-1} \Big(|\pi| \int_{t_i}^{t_{i+1}} \big|(\nabla\Theta)(r)|^2 \mathrm{d}r + \int_{t_i}^{t_{i+1}} |\nabla Z_r|^2 \mathrm{d}r\Big)\Big] \\
&\quad = |\pi| \mathbb{E}\Big[\sup_{0\le t\le T} |(\nabla X_t)^{-1}\sigma(t, X_t)|^2 \Big(|\pi| \int_0^T \big|(\nabla\Theta)(r)|^2 \mathrm{d}r + \int_0^T |\nabla Z_r|^2 \mathrm{d}r\Big)\Big] \\
&\quad \le C|\pi|,
\end{aligned}
$$

where in the last line, we use the fact that $\nabla X, (\nabla X)^{-1}, X \in \mathcal{S}_0^q$ for every $q \ge 2$ and that $\nabla Y, \nabla Z \in \mathcal{H}_0^p$ for some $p > 2$ (in combination with Hölder's inequality) to conclude the finiteness of the expectation. Combining this estimate with those for I_2 and I_3 finishes the proof. □

Towards a time discretization of delay FBSDE

With path regularity for FBSDE with time-delayed generators at hand, one can start discussing a time discretization scheme. Given the nature of this class of BSDE, a time discretization would naturally require some decoupling technique to handle the backward-in-time feature of the equation and the backward-in-time feature of the delay. Applying the backward time discretization from Bouchard and Touzi [24] to (5.33)-(5.34), we obtain for a partition $\pi : 0 = t_0 < t_1 < \ldots < t_N = T$ with step size $\Delta_i = t_{i+1} - t_i$

$$
\begin{aligned}
&Y_{t_N}^\pi = g(X_{t_N}^\pi), \\
&Z_{t_i}^\pi = \mathbb{E}\Big[\frac{W_{t_{i+1}} - W_{t_i}}{\Delta_i} Y_{t_{i+1}}^\pi | \mathcal{F}_{t_i}\Big], \qquad Y_{t_i}^\pi = \mathbb{E}\Big[Y_{t_{i+1}}^\pi | \mathcal{F}_{t_i}\Big] + \Delta_i f(t_i, \Theta_{t_i}^\pi), \\
&\text{where} \quad \Theta_{t_i}^\pi = \Big(\sum_{j=0}^{i} X_{t_j}^\pi \alpha_{\mathcal{X}}\big([t_j, t_{j+1})\big), \sum_{j=0}^{i} Y_{t_j}^\pi \alpha_{\mathcal{Y}}\big([t_j, t_{j+1})\big), \sum_{j=0}^{i} Z_{t_j}^\pi \alpha_{\mathcal{Z}}\big([t_j, t_{j+1})\big)\Big).
\end{aligned}
$$

However, this backward scheme cannot be implemented because in the computation of each $Y_{t_i}^\pi$ running backward from $i = N-1$ to $i = 0$, we must evaluate $\Theta^\pi(t_i)$ which depends on all $Y_{t_j}^\pi$, $Z_{t_j}^\pi$ running in forward direction $j = 0, \ldots, i$.

However, Bender and Denk [13] propose for standard Lipschitz BSDEs a time discretization scheme which mimics the Picard iteration for proving existence and uniqueness of BSDEs. Due to the fact that in each iteration step, one solves an explicit BSDE, the scheme from Bender and Denk [13] runs *forward* in time. The price to pay is to control apart from the error contribution of the time discretization the additional error arising from the Picard iterates (see Theorem 2 in Bender and Denk [13]). We can adapt this idea to (5.33)-(5.34) by exploiting the fact that the solution (Y, Z) is obtained as a limit of (Y^p, Z^p) as p goes infinity. Starting with e.g. $(Y^0, Z^0) = (0,0)$, we get for $p \in \mathbb{N}_0$

$$Y_t^{p+1} = g(X_T) + \int_t^T f\big(s, \Theta^p(s)\big)\mathrm{d}s - \int_t^T Z_s^{p+1}\mathrm{d}W_s, \quad t \in [0,T]$$
$$\text{where} \quad \Theta^p(t) = \Big(\int_{-T}^0 X_{t+v}\alpha_{\mathcal{X}}(\mathrm{d}v), \int_{-T}^0 Y_{t+v}^p \alpha_{\mathcal{Z}}(\mathrm{d}v), \int_{-T}^0 Z_{t+v}^p \alpha_{\mathcal{Z}}(\mathrm{d}v)\Big).$$

The discretization hereof is initiated by $(Y^{\pi,0}, Z^{\pi,0}) = (0,0)$ then constructed iteratively for $p \geq 1$ and $0 \leq i \leq N-1$

$$Y_{t_i}^{\pi,p+1} = \mathbb{E}\Big[g\big(X_{t_N}^\pi\big) + \sum_{j=i}^{N-1} f(t_j, \Theta_{t_j}^{\pi,p})\Delta_j \,\big|\mathcal{F}_{t_i}\Big],$$
$$Z_{t_i}^{\pi,p+1} = \mathbb{E}\Big[\frac{W_{t_{i+1}} - W_{t_i}}{\Delta_i}\Big(g(X_{t_N}^\pi) + \sum_{j=i+1}^{N-1} f(t_j, \Theta_{t_j}^{\pi,p})\Delta_j\Big)\big|\mathcal{F}_{t_i}\Big],$$
$$\text{where} \quad \Theta_{t_i}^{\pi,p} = \Big(\sum_{j=0}^{i} X_{t_j}^\pi \alpha_{\mathcal{X}}\big([t_j, t_{j+1})\big), \sum_{j=0}^{i} Y_{t_j}^{\pi,p}\alpha_{\mathcal{Y}}\big([t_j, t_{j+1})\big), \sum_{j=0}^{i} Z_{t_j}^{\pi,p}\alpha_{\mathcal{Z}}\big([t_j, t_{j+1})\big)\Big).$$

The proof of convergence for this time discretization scheme is left for future research.

6 Dual representations for general multiple stopping problems

In this chapter, we study the dual representation for generalized multiple stopping problems. These representations can be used to solve the pricing problem of general multiple exercise options. We derive a dual representation which allows for cashflows that are subject to volume constraints modeled by integer valued adapted processes and refraction periods modeled by stopping times. Volume constraints cap the simultaneous exercise of several rights. Refraction periods specify the least waiting time that has to elapse once a number of exercise rights have been called. This chapter extends the works by Schoenmakers [119], Bender [11], Bender [12], Aleksandrov and Hambly [1] and Meinshausen and Hambly [93] on the pricing of multiple exercise options, which either take into consideration a refraction period or volume constraints, but not both simultaneously. We stress that to the best of our knowledge, dual representations for multiple exercise options subject to both volume constraints and refraction periods are derived here for the first time. We supplement the theoretical results with an explicit Monte Carlo algorithm for constructing confidence intervals for the price of multiple exercise options and exemplify it by a numerical study on the pricing of a swing option in an electricity market.

6.1 General multiple stopping problem

In this section we consider a multiple stopping problem in discrete time $i = 0, \dots, T$ where $T \in \mathbb{N}$ is a fixed and finite time horizon. Every $j \in \{0, \dots, T\}$ represents an exercise day. We further introduce a "cemetery time" $\partial := T + 1$ where all rights are to be exercised, which have not been exercised before and up to time T. For a given filtration $(\mathcal{F}_i)_{0 \le i \le \partial}$ and a number L of exercise dates we next consider a cashflow X as a map $X : \{0, ..., T, \partial\}^L \times \Omega \to \mathbb{R}$ which satisfies for all $0 \le i_1 \le \cdots \le i_L \le \partial$,

$$X_{i_1,...,i_L} \text{ is } \mathcal{F}_{i_L}\text{-measurable}, \quad \mathbb{E}\,|X_{i_1,...,i_L}| < \infty.$$

With the conventions $\sup := \operatorname{esssup}$, $\mathbb{E}_i := \mathbb{E}_{\mathcal{F}_i}$, let us now consider the stopping problem

$$Y_i^{*L} = \sup_{i \le \tau^1 \le \cdots \le \tau^L} \mathbb{E}_i\, X_{\tau^1,...,\tau^L}, \quad i = 0, \dots, T,$$

where the supremum runs over a family of ordered stopping times $\tau^k, 1 \le k \le L$. Let us define for $k = 2, \dots, L$ and $0 \le j_1 \dots \le j_{k-1} \le r \le \partial$,

$$Y_r^{*L-k+1,j_1,\dots,j_{k-1}} := \sup_{r \le \tau^k \le \dots \le \tau^L} \mathbb{E}_r X_{j_1,\dots,j_{k-1},\tau^k,\dots,\tau^L}, \tag{6.1}$$

with the convention that for $k = 1$, we put $Y_r^{*L,\varnothing} := Y_r^{*L}$, and for $k = L+1$, we put $Y_r^{*0,j_1,\dots,j_L} = X_{j_1,\dots,j_L}$.

Proposition 6.1.1. *We have the following reduction principle*

$$Y_r^{*L-k+1,j_1,\dots,j_{k-1}} = \sup_{\tau \ge r} \mathbb{E}_r Y_\tau^{*L-k,j_1,\dots,j_{k-1},\tau}, \quad r \ge j_{k-1}. \tag{6.2}$$

Proof. This principle can be straightforwardly proved in an inductive manner, but it can also be considered as a discrete time version of a related result in a continuous time setting from Kobylanski et al. [78]. □

Let us give useful a remark.

Remark 6.1.1. *Let $p \in \{0, \dots, T\}$. Given a supermartingale $(Y_r)_{r \ge p}$, we say that a martingale $(M_r)_{r \ge p}$ is a Doob martingale of $(Y_r)_{r \ge p}$, whenever there exists a predictable process $(A_r)_{r \ge p}$, such that $Y_r - M_r + A_r$, for any $r \ge p$, is $\mathcal{F}_p$-measurable. In particular, for any two Doob martingales $(M_r)_{r \ge p}$ and $(\widetilde{M}_r)_{r \ge p}$ of $(Y_r)_{r \ge p}$, it holds*

$$M_r - M_{r'} = \widetilde{M}_r - \widetilde{M}_{r'} = \sum_{k=r'}^{r-1} \left(Y_{k+1} - \mathbb{E}_k Y_{k+1} \right)$$

for any $r \ge r' \ge p$.

We can now state and prove a dual representation for the general multiple stopping problem in terms of martingales.

Theorem 6.1.1 (**Dual representation**). *In the setting described above, we have*

(i) *For any $0 \le i \le \partial$ and any family of martingales $\left(M_r^{L-k+1,j_1,\dots,j_{k-1}} \right)_{r \ge j_{k-1}}$, where $1 \le k \le L$, and $i =: j_0 \le j_1 \le \dots \le j_{k-1}$, we have*

$$Y_i^{*L} \le \mathbb{E}_i \max_{i \le j_1 \le \dots \le j_L \le \partial} \left(X_{j_1,\dots,j_L} + \sum_{k=1}^{L} \left(M_{j_{k-1}}^{L-k+1,j_1,\dots,j_{k-1}} - M_{j_k}^{L-k+1,j_1,\dots,j_{k-1}} \right) \right). \tag{6.3}$$

(ii) *For $i \ge 0$ we have*

$$Y_i^{*L} = \max_{i \le j_1 \le \dots \le j_L \le \partial} \left(X_{j_1,\dots,j_L} + \sum_{k=1}^{L} \left(M_{j_{k-1}}^{*L-k+1,j_1,\dots,j_{k-1}} - M_{j_k}^{*L-k+1,j_1,\dots,j_{k-1}} \right) \right)$$

where for $1 \le k \le L$, and $i =: j_0 \le j_1 \le \dots \le j_{k-1}$, $\left(M_r^{*L-k+1,j_1,\dots,j_{k-1}} \right)_{r \ge j_{k-1}}$ is a Doob martingale of $\left(Y_r^{*L-k+1,j_1,\dots,j_{k-1}} \right)_{r \ge j_{k-1}}$.

Proof. ***(i)*** For the martingale family as stated we have for any chain of stopping times $0 \le \tau^1 \le \cdots \le \tau^L \le \partial$,

$$\begin{aligned}&\mathbb{E}_i \sum_{k=1}^{L} \left(M_{\tau^{k-1}}^{L-k+1,\tau^1,\dots,\tau^{k-1}} - M_{\tau^{k}}^{L-k+1,\tau^1,\dots,\tau^{k-1}} \right) \\ &= \sum_{k=1}^{L} \mathbb{E}_i \mathbb{E}_{\tau^{k-1}} \left(M_{\tau^{k-1}}^{L-k+1,\tau^1,\dots,\tau^{k-1}} - M_{\tau^{k}}^{L-k+1,\tau^1,\dots,\tau^{k-1}} \right) = 0,\end{aligned}$$

hence,

$$Y_i^{*L} = \sup_{i \le \tau^1 \le \cdots \le \tau^L} \mathbb{E}_i \left(X_{\tau^1,\dots,\tau^L} + \sum_{k=1}^{L} \left(M_{\tau^{k-1}}^{L-k+1,\tau^1,\dots,\tau^{k-1}} - M_{\tau^{k}}^{L-k+1,\tau^1,\dots,\tau^{k-1}} \right) \right)$$

from which (i) follows directly.

(ii) For any chain $i \le j_1 \le \cdots \le j_L \le \partial$, an application of the Doob decomposition yields (recalling $j_0 := i$)

$$\begin{aligned}&X_{j_1,\dots,j_L} + \sum_{k=1}^{L} \left(M_{j_{k-1}}^{*L-k+1,j_1,\dots,j_{k-1}} - M_{j_k}^{*L-k+1,j_1,\dots,j_{k-1}} \right) \\ &= X_{j_1,\dots,j_L} + \sum_{k=1}^{L} \left(Y_{j_{k-1}}^{*L-k+1,j_1,\dots,j_{k-1}} - Y_{j_k}^{*L-k+1,j_1,\dots,j_{k-1}} \right) \\ &+ \sum_{k=1}^{L} \sum_{l=j_{k-1}}^{j_k - 1} \left(\mathbb{E}_l Y_{l+1}^{*L-k+1,j_1,\dots,j_{k-1}} - Y_l^{*L-k+1,j_1,\dots,j_{k-1}} \right) \\ &= Y_i^{*L} + \sum_{k=1}^{L} \left(Y_{j_k}^{*L-k,j_1,\dots,j_k} - Y_{j_k}^{*L-k+1,j_1,\dots,j_{k-1}} \right) \\ &+ \sum_{k=1}^{L} \sum_{l=j_{k-1}}^{j_k - 1} \left(\mathbb{E}_l Y_{l+1}^{*L-k+1,j_1,\dots,j_{k-1}} - Y_l^{*L-k+1,j_1,\dots,j_{k-1}} \right). \qquad (6.4)\end{aligned}$$

By the reduction principle (6.2) it follows that $\left(Y_r^{*L-k+1,j_1,\dots,j_{k-1}} \right)_{r \ge j_{k-1}}$ is a supermartingale which dominates the (virtual) cashflow $Y_r^{*L-k,j_1,\dots,j_{k-1},r}$ for $k = 1, \dots, L$. Hence, expression (6.4) is less than or equal to Y_i^{*L}. It thus follows that

$$\max_{i \le j_1 \le \cdots \le j_L \le \partial} \left(X_{j_1,\dots,j_L} + \sum_{k=1}^{L} \left(M_{j_{k-1}}^{*L-k+1,j_1,\dots,j_{k-1}} - M_{j_k}^{*L-k+1,j_1,\dots,j_{k-1}} \right) \right) \le Y_i^{*L},$$

and then, an application of (i) finishes the proof. □

A straightforward consequence of Theorem 6.1.1 is the following dual representation in terms of approximate Snell envelopes.

Corollary 6.1.1. *For any family* $\left(Y_r^{L-k+1,j_1,\ldots,j_{k-1}}\right)_{r\ge j_{k-1}}$ *of approximations to the Snell envelopes* $\left(Y_r^{*L-k+1,j_1,\ldots,j_{k-1}}\right)_{r\ge j_{k-1}}$ *with* $Y_{j_L}^{0,j_1,\ldots,j_L} := X_{j_1,\ldots,j_L}$, *we have for* $i \ge 0$

$$\begin{aligned} Y_i^{*L} \le Y_i^L + \mathbb{E}_i \max_{i\le j_1\le\cdots\le j_L\le\partial} \sum_{k=1}^{L} \Bigg(& Y_{j_k}^{L-k,j_1,\ldots,j_k} - Y_{j_k}^{L-k+1,j_1,\ldots,j_{k-1}} \\ & + \sum_{l=j_{k-1}}^{j_k-1} \left(\mathbb{E}_l Y_{l+1}^{L-k+1,j_1,\ldots,j_{k-1}} - Y_l^{L-k+1,j_1,\ldots,j_{k-1}}\right)\Bigg). \end{aligned} \tag{6.5}$$

Equality holds when the Snell envelopes are plugged in.

Proof. Given $\left(Y_r^{L-k+1,j_1,\ldots,j_{k-1}}\right)_{r\ge j_{k-1}}$, we denote a corresponding family of Doob martingales by $\left(M_r^{L-k+1,j_1,\ldots,j_{k-1}}\right)_{r\ge j_{k-1}}$. Following the same manipulations as in (6.4) and recalling that by definition we have $Y_{j_L}^{0,j_1,\ldots,j_L} = X_{j_1,\ldots,j_L}$, we get

$$\begin{aligned} & Y_i^L + \sum_{k=1}^{L} \Bigg(\left(Y_{j_k}^{L-k,j_1,\ldots,j_k} - Y_{j_k}^{L-k+1,j_1,\ldots,j_{k-1}}\right) \\ & \quad + \sum_{l=j_{k-1}}^{j_k-1} \left(\mathbb{E}_l Y_{l+1}^{L-k+1,j_1,\ldots,j_{k-1}} - Y_l^{L-k+1,j_1,\ldots,j_{k-1}}\right)\Bigg) \\ & = X_{j_1,\ldots,j_L} + \sum_{k=1}^{L} \left(M_{j_{k-1}}^{L-k+1,j_1,\ldots,j_{k-1}} - M_{j_k}^{L-k+1,j_1,\ldots,j_{k-1}}\right). \end{aligned}$$

The claim follows by a straightforward reformulation of Theorem 6.1.1. □

We shall point out that the dual representation from Theorem 6.1.1 is in terms of families of martingales $\left(M_r^{L-k+1,j_1,\ldots,j_{k-1}}\right)_{r\ge j_{k-1}}$ whose size is parameterized by the ordered $(k-1)$-tuples $(j_1,\ldots,j_{k-1}), k = 1,\ldots,L$. Depending on the size of T and the number of exercise rights L a huge number of martingales $M^{L-k+1,j_1,\ldots,j_{k-1}}$, $k = 1,\ldots,L$, $0 \le j_1 \le \ldots \le j_{k-1} \le \partial = T+1$ may be required to compute an upper price bound by means of the dual formulation above. Thus, bearing in mind that one wishes to implement this dual representation, it is of great importance to single out situations, in which a family of optimal martingales can be constructed from a much smaller family of auxiliary processes. This is the topic of Section 6.2. A motivating example in this respect is the standard multiple stopping problem.

Example 6.1.1 (Standard multiple stopping). *Let* Z_j *be a non-negative adapted process such that* $Z_j = 0$ *for* $j = \partial$, *i.e. no penalty is imposed for unexercised rights. The standard multiple stopping problem is to maximize* $\mathbb{E}[\sum_{k=1}^{L} Z_{\tau^k}]$ *over the set of ordered stopping times* $\tau^1 \le \cdots \le \tau^L$ *such that* $\tau^k < \tau^{k+1}$ *or* $\tau^k = \tau^{k+1} = \partial$. *This means that at most one right can be exercised at each of the exercise days* $j \in \{0,\ldots,T\}$, *but an arbitrary number of rights can be left unexercised. We handle the case of not exercising by the convention that all rights that have not been exercised up to maturity* T *have to* *be exercised at the cemetery time ∂ (which here entails penalty payments of zero, i.e. no penalty payments). This problem can be formulated in our general setting by considering the cashflow*

$$X_{i_1,\dots,i_L} = \begin{cases} \sum_{k=1}^{L} Z_{i_k}, & \text{if } i_{j+1} = i_j \Rightarrow i_j = \partial\,, \\ -N, & \text{else }\;, \end{cases}$$

for $N \in \mathbb{N}$, meaning that if two successive exercise days collapse to one, $i_{j+1} = i_j$, then these are unexercised rights which one gets rid off at cemetery time. Here, $-N$ denotes a penalty payment which becomes active once any of the exercise rules are violated. Note that the Snell envelope Y_i^{*L} does not depend on the choice of N, because it is never optimal to exercise X in a way which yields a negative cashflow. Hence, letting N tend to ∞, for an ordered L-tuple $i \le j_1 \le \cdots \le j_L$ satisfying $j_k = j_{k+1} \Rightarrow j_k = \partial$, Theorem 6.1.1 yields

$$Y_i^{*L} = \max_{\substack{i \le j_1 \le \cdots \le j_L \\ j_k = j_{k+1} \Rightarrow j_k = \partial}} \left(\sum_{k=1}^{L} Z_{j_k} + \sum_{k=1}^{L} \left(M_{j_{k-1}}^{*L-k+1,j_1,\dots,j_{k-1}} - M_{j_k}^{*L-k+1,j_1,\dots,j_{k-1}} \right) \right)$$

where for $1 \le k \le L$, $i \le j_1 < \cdots < j_{k-1}$, $\left(M_r^{*L-k+1,j_1,\dots,j_{k-1}} \right)_{r \ge j_{k-1}}$ is a Doob martingale of

$$Y_r^{*L-k+1,j_1,\dots,j_{k-1}} = \sum_{p=1}^{k-1} Z_{j_p} + \sup_{\substack{r \le \tau^k \le \cdots \le \tau^L \\ \tau^p = \tau^{p+1} \Rightarrow \tau^p = \partial \text{ and } \tau^k = j_{k-1} \Rightarrow j_{k-1} = \partial}} \mathbb{E}_r \sum_{p=k}^{L} Z_{\tau^p}.$$

for $r \ge j_{k-1}$. Define for $r \ge 0$

$$Y_r^{*L-k+1} := \sup_{\substack{r \le \tau^k \cdots \le \tau^L \\ \tau^p = \tau^{p+1} \Rightarrow \tau^p = \partial}} \mathbb{E}_r \left(\sum_{p=k}^{L} Z_{\tau^p} \right),$$

and denote the Doob martingale of $(Y_r^{*L-k+1})_{r \ge 0}$ by $(M_r^{*L-k+1})_{r \ge 0}$. Since

$$Y_r^{*L-k+1} - Y_r^{*L-k+1,j_1,\dots,j_{k-1}}$$

is $\mathcal{F}_{j_{k-1}}$-measurable for $r \ge j_{k-1}$, we can conclude by Remark 6.1.1 that $(M_r^{*L-k+1})_{r \ge j_{k-1}}$ is a Doob martingale of $\left(Y_r^{*L-k+1,j_1,\dots,j_{k-1}} \right)_{r \ge j_{k-1}}$. Hence we end up with the dual representation

$$Y_i^{*L} = \max_{\substack{i \le j_1 \le \cdots \le j_L \\ j_k = j_{k+1} \Rightarrow j_k = \partial}} \left(\sum_{k=1}^{L} Z_{j_k} + \sum_{k=1}^{L} \left(M_{j_{k-1}}^{*L-k+1} - M_{j_k}^{*L-k+1} \right) \right)$$

of Schoenmakers [119]. Here, the potentially very large family of optimal martingales $\left(M^{*L-k+1,j_1,\dots,j_{k-1}} \right)$, $k = 1, \dots, L$, $0 \le j_1, \dots \le j_{k-1} \le \partial$, collapses to a family of L martingales, namely the Doob martingales of Y^{*k}.

6.2 Generic cashflow with additive and multiplicative structure

We now introduce a generic cashflow structure for which the dual representation simplifies in a similar way as for the standard multiple stopping problem in Example 6.1.1. To this end, let us consider for each $k = 1, ..., L$ and $l = 1, ..., L-1$ two adapted processes U^k and V^l. We define a "pre-cashflow"

$$\widetilde{X}_{j_1,...,j_L} = \sum_{k=1}^{L} U^k_{j_k} \prod_{l=1}^{k-1} V^l_{j_l},$$

which is assumed to satisfy $\widetilde{X}_{j_1,...,j_L} > -N$ for some (possibly large) $N \in \mathbb{N}$. Concerning the processes U^k and V^l, we suppose that U^k_i is integrable for every $k = 1, \ldots, L$ and $i = 0, \ldots, \partial$, and that V^l_i is strictly positive and bounded from above for every $l = 1, \ldots, L-1$ and $i = 0, \ldots, \partial$. The multiple stopping problem which we have in mind is to optimally exercise this pre-cashflow under some constraints on the set of admissible stopping times, which we now formulate. We first define a volume constraint process v_j that is adapted and takes its values in $\{1, ..., L\}$ such that v_j is the maximum number of rights one may exercise at j, and such that $v_\partial = L$. In order to formalize this constraint, we introduce for $p \geq 1$ the mapping $\mathcal{E}_p$ which acts on a non-decreasing p-tuple $(j_1, \ldots, j_p)$ by

$$\mathcal{E}_p(j_1, ..., j_p) := \#\{r : 1 \leq r \leq p,\ j_r = j_p\}.$$

Hence, $\mathcal{E}_p$ denotes the number of rights exercised at j_p in the non-decreasing chain $0 \leq j_1 \leq \cdots \leq j_p \leq \partial$. Obviously, an ordered chain of stopping times $\tau^1 \leq \cdots \leq \tau^L$ satisfies the volume constraint if and only if $\mathcal{E}_p(\tau^1, ..., \tau^p) \leq v_{\tau^p}$ for every $p = 1, \ldots, L$. The second constraint is a refraction period which specifies the minimal waiting time between two exercises at different times. We admit random refraction periods, i.e. at each time i, $0 \leq i < \partial$, we fix a stopping time ρ^i taking values in $\{i+1, ..., \partial\}$. If at least one right is exercised at time i, then the refraction period constraint imposes that the next right must either be exercised at the same time (if consistent with the volume constraint) or otherwise no earlier than ρ^i. A standard case is $\rho^i = (i+\delta) \wedge \partial$, where $1 \leq \delta \leq T$ is deterministic. Both constraints can be summarized by the binary $\mathcal{F}_{j_p}$-measurable random variable

$$\mathcal{C}_p(j_1, ..., j_p) := \begin{cases} 1, & \forall_{1 \leq l \leq p} :\ \mathcal{E}_l(j_1, ..., j_l) \leq v_{j_l} \text{ and } \forall_{1 \leq l \leq p} :\ j_l > j_{l-1} \Longrightarrow j_l \geq \rho^{j_{l-1}} \\ 0, & \text{else,} \end{cases}$$

for the ordered p-tuple $j_1, ..., j_p$ with values in $\{0, \ldots, T\}$. Now $\mathcal{C}_p(j_1, ..., j_p)$ is equal to 1 if and only if the constraints are satisfied when exercising p-times at $j_1 \leq \cdots \leq j_p$. The dynamic multiple stopping problem which we now study is

$$Y^{*L}_i = \sup_{\substack{i \leq \tau^1 \leq \cdots \leq \tau^L \leq \partial \\ \mathcal{C}_L(\tau^1, ..., \tau^L) = 1}} \mathbb{E}_i \left[\sum_{k=1}^{L} U^k_{\tau^k} \prod_{l=1}^{k-1} V^l_{\tau^l}, \right] \tag{6.6}$$

i.e. the supremum is taken over all stopping times with values in $\{i,\ldots,T,\partial\}$ which satisfy the volume and the refraction period constraints. This problem fits into our general (unconstrained) setting by considering the cashflow

$$X_{j_1,\ldots,j_L} = \begin{cases} \widetilde{X}_{j_1,\ldots,j_L}, & \text{if } \mathcal{C}_L(j_1,,\ldots,j_L) = 1, \\ -N, & \text{else.} \end{cases} \tag{6.7}$$

To illustrate our motivation for studying such cashflows, let us have a look at the following examples.

Example 6.2.1 (**Swing options**). *We extend the situation in Example 6.1.1 by imposing volume constraints and refraction periods as described above. Hence, we have*

$$\begin{aligned} V_j^l &:= 1,\ l = 1,\ldots,L-1,\ j = 0,\ldots,\partial, \\ U_j^p &:= Z_j\ p = 1,\ldots,L,\ j = 0,\ldots,\partial, \end{aligned}$$

where we recall that Z is a non-negative adapted process with $Z_\partial = 0$. The multiple stopping problem then becomes

$$\sup_{\substack{\tau^1\le\cdots\le\tau^L \\ \mathcal{C}_L(\tau^1,\ldots,\tau^L)=1}} \mathbb{E}\left[\sum_{k=1}^{L} Z_{\tau^k}\right],$$

leading to

$$X_{j_1,\ldots,j_L} = \begin{cases} \sum_{k=1}^L Z_{j_k}, & \textit{if } \mathcal{C}_L(j_1,,\ldots,j_L) = 1, \\ -N, & \textit{else.} \end{cases}$$

Here, any $N \in \mathbb{N}$ can be chosen because Z is non-negative. A dual approach for this multiple stopping problem was studied by Bender [12] and Aleksandrov and Hambly [1] under volume constraints, but with unit refraction period, i.e. $\rho^i = i+1$. The case of non-trivial constant refraction period is treated in Bender [11], but only under unit volume constraint, i.e. $v_i = 1$. A typical problem in the context of electricity markets which leads to this type of multiple stopping problem is the pricing of swing option contracts, in which volume constraints and refraction periods are often imposed. This option pricing problem will be explained in more detail in our numerical study in Section 6.3.

Example 6.2.2 (**Exponential utility**). *Under the assumptions of the previous example we can also maximize the exponential utility of exercising the cashflow Z_i L-times while obeying the constraints. Given a risk aversion parameter $\alpha \in (0,\infty)$ the corresponding multiple stopping problem becomes*

$$\sup_{\substack{\tau^1\le\cdots\le\tau^L \\ \mathcal{C}_L(\tau^1,\ldots,\tau^L)=1}} \mathbb{E}\left[-e^{-\alpha \sum_{k=1}^L Z_{\tau^k}}\right].$$

This problem fits in our setting by considering

$$X_{j_1,\ldots,j_L} = \begin{cases} \sum_{k=1}^L U^k_{j_k} \prod_{l=1}^{k-1} V^l_{j_l},, & \text{if } \mathcal{C}_L(j_1,,\ldots,j_L) = 1, \\ -N, & \text{else} \end{cases}$$

with

$$V^l_j := e^{-\alpha Z_j} > 0 \quad \text{and} \quad U^k_j := \begin{cases} 0, & \text{if } k = 1,\ldots,L-1, \\ -e^{-\alpha Z_j}, & \text{if } k = L, \end{cases}$$

for $j = 0,1,\ldots,\partial$ *and* $N \geq 2$.

Example 6.2.3 (**Portfolio liquidation**). *Suppose a (large) investor on a illiquid market wants to sell out (liquidate) L shares of a stock during the period $\{0,\ldots,T\}$. We assume that $\widetilde{S}_j > 0$, $j = 0,\ldots,T$, is the virtual stock price process reflecting the stock price evolution in the absence of the large investor's trading. In the spirit of Section 3.1 from Schied and Slynko [118], we model the price impact of the large investor by a resilience function G which we here apply to the log-price. Hence, the log-stock price $\ln S^{j_1,\ldots,j_{k-1}}_{j_k}$ at time j_k of the sale of the k-th share, where $k-1$ shares were already sold at dates $0 \leq j_1 \leq \cdots \leq j_{k-1}$, is given by*

$$\ln S^{j_1,\ldots,j_{k-1}}_{j_k} = \ln \widetilde{S}_{j_k} - \sum_{l=1}^{k-1} G(j_k - j_l).$$

We here choose the capped linear resilience function $G(t) = b(1-at)_+$ for constants $a,b > 0$. Assuming a short time horizon $T \leq 1/a$, the investor is thus faced with a multiple stopping problem

$$\sup_{\tau^1 \leq \cdots \leq \tau^L} \mathbb{E}\left[\sum_{k=1}^L S^{\tau^1,\ldots,\tau^{k-1}}_{\tau^k}\right],$$

which fits in our framework by applying, for $0 \leq j_1 \leq \cdots \leq j_L \leq T$, the cashflow

$$\begin{aligned} X_{j_1,\ldots,j_L} &:= \sum_{k=1}^L S^{j_1,\ldots,j_{k-1}}_{j_k} = \sum_{k=1}^L \widetilde{S}_{j_k} \exp\left(-\sum_{l=1}^{k-1} b\left(1 - a\left(j_k - j_l\right)\right)\right) \\ &= \sum_{k=1}^L \widetilde{S}_{j_k} \exp\left[b\left(a\, j_k - 1\right)\left(k-1\right)\right] \prod_{l=1}^{k-1} \exp\left(-abj_l\right) \\ &= \sum_{k=1}^L U^k_{j_k} \prod_{l=1}^{k-1} V^l_{j_l} \end{aligned}$$

with

$$U^k_j := \widetilde{S}_j \exp\left[b\left(aj-1\right)\left(k-1\right)\right] \quad \text{and} \quad V^l_j := \exp\left(-abj\right).$$

Note that the cemetery time ∂ is irrelevant in this setting and we can e.g. set $U^k_\partial = 0$, $V^l_\partial = 1$ to make sure that it is never optimal to exercise at this time.

Similar to the situation in Example 6.1.1, we now introduce a family of auxiliary multiple stopping problems Y_r^{*L-k+1}, which are not parameterized by the times $j_1, \dots, j_{k-1}$, at which the first rights were exercised. We will then show that a family of optimal martingales for the original multiple stopping problem (6.6) can be constructed via the Doob decomposition of the auxiliary problems Y_r^{*L-k+1}. This then leads to a simplified dual representation for (6.6), which can be implemented in practice even when the maturity T and the number of rights L are large. Define

$$Y_r^{*L-k+1} := \sup_{\substack{\tau^k,\cdots,\tau^L \\ r\le\tau^k\le\cdots\le\tau^L \text{ and } \mathcal{C}_{L-k+1}(\tau^k,\dots,\tau^L)=1}} \mathbb{E}_r\left(\sum_{p=k}^{L} U_{\tau^p}^p \prod_{l=k}^{p-1} V_{\tau^l}^l\right), \tag{6.8}$$

with the convention $Y_r^{*0} := 0$. The following result states the Bellman principle for this multiple stopping problem.

Proposition 6.2.1 (Dynamic program). *For $r \ge 0$ and $1 \le k \le L$, we have*

$$Y_r^{*L-k+1} = \max\left(\mathbb{E}_r Y_{r+1}^{*L-k+1}, \max_{1\le n\le v_r \wedge (L-k+1)}\left(\sum_{p=k}^{k+n-1} U_r^p \prod_{l=k}^{p-1} V_r^l + \prod_{l=k}^{k+n-1} V_r^l \mathbb{E}_r Y_{\rho^r}^{*L-k-n+1}\right)\right).$$

Proof. Let $0 \le n \le v_r \wedge (L-k+1)$ satisfy $\tau^k = \dots \tau^{k+n-1} = r$ and $\tau^{k+n} > r$. Then (6.8) can be written as

$$\begin{aligned}
&Y_r^{*L-k+1} \\
&= \max_{0\le n\le v_r\wedge(L-k+1)} \sup_{\substack{r<\tau^{k+n}\le\cdots\le\tau^L \\ \mathcal{C}_{L-k+1}(r,\dots,r,\tau^{k+n},\dots,\tau^L)=1}} \mathbb{E}_r\left(\sum_{p=k}^{k+n-1} U_r^p \prod_{l=k}^{p-1} V_r^l + \sum_{p=k+n}^{L} U_{\tau^p}^p \prod_{l=k}^{k+n-1} V_r^l \prod_{l=k+n}^{p-1} V_{\tau^l}^l\right) \\
&= \max\left(\mathbb{E}_r Y_{r+1}^{*L-k+1}, \max_{1\le n\le v_r\wedge(L-k+1)}\left(\sum_{p=k}^{k+n-1} U_r^p \prod_{l=k}^{p-1} V_r^l\right.\right. \\
&\quad \left.\left. + \prod_{l=k}^{k+n-1} V_r^l \sup_{\substack{\rho^r\le\tau^{k+n}\le\cdots\le\tau^L \\ \mathcal{C}_{L-k-n+1}(\tau^{k+n},\dots,\tau^L)=1}} \mathbb{E}_r\Big(\sum_{p=k+n}^{L} U_{\tau^p}^p \prod_{l=k+n}^{p-1} V_{\tau^l}^l\Big)\right)\right).
\end{aligned}$$

The term $\mathbb{E}_r Y_{r+1}^{*L-k+1}$ arises by putting $n = 0$ and the fact that

$$\begin{aligned}
&\sup_{\substack{r\le\tau^k\le\cdots\le\tau^L \\ \mathcal{C}_{L-k+1}(\tau^k,\dots,\tau^L)=1}} \mathbb{E}_r\left(\sum_{p=k}^{L} U_{\tau^p}^p \prod_{l=k}^{p-1} V_{\tau^l}^l\right) = \sup_{\substack{r\le\tau^k\le\cdots\le\tau^L \\ \mathcal{C}_{L-k+1}(\tau^k,\dots,\tau^L)=1}} \mathbb{E}_r\mathbb{E}_{r+1}\left(\sum_{p=k}^{L} U_{\tau^p}^p \prod_{l=k}^{p-1} V_{\tau^l}^l\right) \\
&\quad = \mathbb{E}_r \sup_{\substack{r\le\tau^k\le\cdots\le\tau^L \\ \mathcal{C}_{L-k+1}(\tau^k,\dots,\tau^L)=1}} \mathbb{E}_{r+1}\left(\sum_{p=k}^{L} U_{\tau^p}^p \prod_{l=k}^{p-1} V_{\tau^l}^l\right) = \mathbb{E}_r Y_{r+1}^{*L-k+1},
\end{aligned}$$

because the set over which the supremum runs is $\mathcal{F}_r$-measurable. An analogous argument shows that

$$\begin{aligned}
&\sup_{\substack{\rho^r \le \tau^{k+n} \le \cdots \le \tau^L \\ \mathcal{C}_{L-k-n+1}(\tau^{k+n},\ldots,\tau^L)=1}} \mathbb{E}_r\Big(\sum_{p=k+n}^{L} U^p_{\tau^p} \prod_{l=k+n}^{p-1} V^l_{\tau^l}\Big) \\
&= \mathbb{E}_r \sup_{\substack{\rho^r \le \tau^{k+n} \le \cdots \le \tau^L \\ \mathcal{C}_{L-k-n+1}(\tau^{k+n},\ldots,\tau^L)=1}} \mathbb{E}_{\rho^r}\Big(\sum_{p=k+n}^{L} U^p_{\tau^p} \prod_{l=k+n}^{p-1} V^l_{\tau^l}\Big) = \mathbb{E}_r Y^{*L-k-n+1}_{\rho^r},
\end{aligned} \tag{6.9}$$

and this concludes the proof. □

We now establish a crucial relationship between the Snell envelopes $Y^{*L-k+1,j_1,\ldots,j_{k-1}}$ and Y^{*L-k+1} defined in (6.8). The following Proposition shows that $Y^{*L-k+1,j_1,\ldots,j_{k-1}}$, parameterized by the j_k's, can be represented in terms of Y^{*L-k+1} which avoids the j_k's. Notice that for $k = 1$, both Snell envelopes coincide by definition.

Proposition 6.2.2. *Assume $1 < k \le L+1$ and let $j_1 \le \ldots \le j_{k-1}$ be an ordered tuple taking values in $\{0,\ldots,T\}$. Under the (random) condition $\mathcal{C}_{k-1}(j_1,\ldots,j_{k-1}) = 1$, we have*

(i) for $r > j_{k-1}$

$$Y^{*L-k+1,j_1,\ldots,j_{k-1}}_r = \sum_{p=1}^{k-1} U^p_{j_p} \prod_{l=1}^{p-1} V^l_{j_l} + \mathbb{E}_r Y^{*L-k+1}_{\rho^{j_{k-1}\vee r}} \prod_{l=1}^{k-1} V^l_{j_l}. \tag{6.10}$$

(ii) For $r = j_{k-1}$, we have

$$\begin{aligned}
Y^{*L-k+1,j_1,\ldots,j_{k-1}}_{j_{k-1}} &= \sum_{p=1}^{k-1} U^p_{j_p} \prod_{l=1}^{p-1} V^l_{j_l} \\
&+ \prod_{l=1}^{k-1} V^l_{j_l} \max_{n\in N(j_1,\ldots,j_{k-1})} \left\{ \sum_{p=k}^{k-1+n} U^p_{j_{k-1}} \prod_{l=k}^{p-1} V^l_{j_{k-1}} + \mathbb{E}_{j_{k-1}} Y^{*L-k+1-n}_{\rho^{j_{k-1}}} \prod_{l=k}^{k-1+n} V^l_{j_{k-1}} \right\},
\end{aligned} \tag{6.11}$$

where the maximum runs over the $\mathcal{F}_{j_{k-1}}$-measurable set

$$N(j_1,\ldots,j_{k-1}) := \{n;\ 0 \le n \le (v_{j_{k-1}} - \mathcal{E}_{k-1}(j_1,\ldots,j_{k-1})) \wedge (L-k+1)\}.$$

Proof. For $k = L+1$ both assertions are implied by the conventions $Y^{*0,j_1,\ldots,j_L}_r = X_{j_1,\ldots,j_L}$ and $Y^{*0}_r = 0$. Hence, we assume for the remainder of the proof that $1 < k \le L$.

(i) Under $\mathcal{C}_{k-1}(j_1,\ldots,j_{k-1}) = 1$, we have for $r > j_{k-1}$

$$\begin{aligned}
&Y^{*L-k+1,j_1,\ldots,j_{k-1}}_r \\
&= \sum_{p=1}^{k-1} U^p_{j_p} \prod_{l=1}^{p-1} V^l_{j_l} + \prod_{l=1}^{k-1} V^l_{j_l} \sup_{\substack{r \le \tau^k \le \cdots \le \tau^L \\ \mathcal{C}_L(j_1,\ldots,j_{k-1},\tau^k,\ldots,\tau^L)=1}} \mathbb{E}_r \left(\sum_{p=k}^{L} U^p_{\tau^p} \prod_{l=k}^{p-1} V^l_{\tau^l} \right).
\end{aligned} \tag{6.12}$$

As $r > j_{k-1}$, we obtain, thanks to (6.9),

$$
\begin{aligned}
&\sup_{\substack{r\le\tau^k\le\cdots\le\tau^L\\ \mathcal{C}_L(j_1,\dots,j_{k-1},\tau^k,\dots,\tau^L)=1}} \mathbb{E}_r\left(\sum_{p=k}^{L} U^p_{\tau^p}\prod_{l=k}^{p-1}V^l_{\tau^l}\right)\\
&= \mathbb{1}_{\{r<\rho^{j_{k-1}}\}} \sup_{\substack{\rho^{j_{k-1}}\le\tau^k\le\cdots\le\tau^L\\ \mathcal{C}_{L-k+1}(\tau^k,\dots,\tau^L)=1}} \mathbb{E}_r\left(\sum_{p=k}^{L} U^p_{\tau^p}\prod_{l=k}^{p-1}V^l_{\tau^l}\right)\\
&\quad + \mathbb{1}_{\{r\ge\rho^{j_{k-1}}\}} \sup_{\substack{r\le\tau^k\le\cdots\le\tau^L\\ \mathcal{C}_{L-k+1}(\tau^k,\dots,\tau^L)=1}} \mathbb{E}_r\left(\sum_{p=k}^{L} U^p_{\tau^p}\prod_{l=k}^{p-1}V^l_{\tau^l}\right)\\
&= \mathbb{1}_{\{r<\rho^{j_{k-1}}\}}\mathbb{E}_r Y^{*L-k+1}_{\rho^{j_{k-1}}} + \mathbb{1}_{\{r\ge\rho^{j_{k-1}}\}} Y^{*L-k+1}_{r}. \qquad (6.13)
\end{aligned}
$$

Hence, by combining (6.12) and (6.13) we get (i).

(ii) Given that the first $(k-1)$ rights have been exercised at times $j_1 \le \cdots \le j_{k-1}$, the number of the remaining $(L-k+1)$ rights which are also exercised at time j_{k-1} must be chosen from the $\mathcal{F}_{j_{k-1}}$-measurable set $N(j_1,\dots,j_{k-1}) = \big\{n:\ 0\le n\le (v_{j_{k-1}} - \mathcal{E}_{k-1}(j_1,\dots,j_{k-1}))\wedge(L-k+1)\big\}$. These are the only choices which obey the volume constraint at time j_{k-1}. Hence,

$$
Y^{*L-k+1,j_1,\dots,j_{k-1}}_{j_{k-1}} = \max_{n\in N(j_1,\dots,j_{k-1})} \sup_{\substack{\rho^{j_{k-1}}\le\tau^{n+k}\le\cdots\le\tau^L\\ \mathcal{C}_{L-k-n+1}(\tau^{k+n},\dots,\tau^L)=1}} \mathbb{E}_{j_{k-1}} X_{j_1,\dots,j_{k-1},\cdots,j_{k-1},\tau^{k+n},\dots,\tau^L},
$$

where the time index j_{k-1} appears $(n+1)$-times. We then get for fixed $n\in N(j_1,\dots,j_{k-1})$,

$$
\begin{aligned}
&\sup_{\substack{\rho^{j_{k-1}}\le\tau^{n+k}\le\cdots\le\tau^L\\ \mathcal{C}_{L-k-n+1}(\tau^{k+n},\dots,\tau^L)=1}} \mathbb{E}_{j_{k-1}} X_{j_1,\dots,j_{k-1},\cdots,j_{k-1},\tau^{k+n},\dots,\tau^L}\\
&= \sum_{p=1}^{k-1} U^p_{j_p}\prod_{l=1}^{p-1}V^l_{j_l} + \prod_{l=1}^{k-1}V^l_{j_l}\sum_{p=k}^{k-1+n} U^p_{j_{k-1}}\prod_{l=k}^{p-1}V^l_{j_{k-1}}\\
&\quad + \prod_{l=1}^{k-1}V^l_{j_l}\prod_{l=k}^{k-1+n}V^l_{j_{k-1}} \sup_{\substack{\rho^{j_{k-1}}\le\tau^{n+k}\le\cdots\le\tau^L\\ \mathcal{C}_{L-k-n+1}(\tau^{k+n},\dots,\tau^L)=1}} \mathbb{E}_{j_{k-1}}\sum_{p=k+n}^{L} U^p_{\tau^p}\prod_{l=k+n}^{p-1}V^l_{\tau^p}\\
&= \sum_{p=1}^{k-1} U^p_{j_p}\prod_{l=1}^{p-1}V^l_{j_l} + \prod_{l=1}^{k-1}V^l_{j_l}\sum_{p=k}^{k-1+n} U^p_{j_{k-1}}\prod_{l=k}^{p-1}V^l_{j_{k-1}} + \prod_{l=1}^{k-1}V^l_{j_l}\prod_{l=k}^{k-1+n}V^l_{j_{k-1}}\mathbb{E}_{j_{k-1}}Y^{*L-k-n+1}_{\rho^{j_{k-1}}},
\end{aligned}
$$

making again use of (6.9). This implies (6.11). $\square$

6.2.1 Dual representation based on Doob decompositions

The goal of this subsection is to prove and discuss the following simplified version of the dual representation from Theorem 6.1.1 for multiple stopping problems of the form (6.6). Note in contrast to Theorem 6.1.1, the dual representation gets by with a family of input martingales whose size is reduced to L. This is particularly appealing for numerical implementations.

Theorem 6.2.1. *Suppose Y_i^{*L} is given by (6.6). Then we have the following assertions:*
(i) *For any set of martingales $\left(M_r^{L-k+1}\right)_{r\geq 0}$, $k=1,\ldots,L$, and any set of integrable adapted processes $\left(A_r^{L-k+1}\right)_{r\geq 0}$, $k=1,\ldots,L$, we have for $i\geq 0$ and with $j_0 := i$*

$$Y_i^{*L} \leq \mathbb{E}_i \max_{\substack{i\leq j_1\leq\cdots\leq j_L\leq\partial \\ \mathcal{C}_L(j_1,\ldots,j_L)=1}} \left(\sum_{k=1}^{L} U_{j_k}^k \prod_{l=1}^{k-1} V_{j_l}^l + \sum_{k=1}^{L}\prod_{l=1}^{k-1} V_{j_l}^l \left(M_{j_{k-1}}^{L-k+1} - M_{j_k}^{L-k+1} + \mathbb{E}_{j_k} A_{\rho^{j_{k-1}}}^{L-k+1} - \mathbb{E}_{j_{k-1}} A_{\rho^{j_{k-1}}}^{L-k+1} \right) \right).$$

(ii) *For every $i\geq 0$ with $j_0 := i$, we have*

$$Y_i^{*L} = \max_{\substack{i\leq j_1\leq\cdots\leq j_L\leq\partial \\ \mathcal{C}_L(j_1,\ldots,j_L)=1}} \left(\sum_{k=1}^{L} U_{j_k}^k \prod_{l=1}^{k-1} V_{j_l}^l + \sum_{k=1}^{L}\prod_{l=1}^{k-1} V_{j_l}^l \left(M_{j_{k-1}}^{*L-k+1} - M_{j_k}^{*L-k+1} + \mathbb{E}_{j_k} A_{\rho^{j_{k-1}}}^{*L-k+1} - \mathbb{E}_{j_{k-1}} A_{\rho^{j_{k-1}}}^{*L-k+1} \right) \right),$$

where M^{*L-k+1} , A^{*L-k+1} are the martingale part and the predictable part of the Doob decomposition of the auxiliary Snell envelopes Y^{*L-k+1} in (6.8), respectively.

Let us recall that the Doob decomposition of Y^{*L-k+1} is the unique decomposition of the form

$$Y_r^{*L-k+1} = Y_0^{*L-k+1} + M_r^{*L-k+1} - A_r^{*L-k+1},$$

where the martingale M_r^{*L-k+1} and the predictable process A_r^{*L-k+1} start in zero at time zero. In order to prove Theorem 6.2.1 we need the following auxiliary result.

Proposition 6.2.3. *Under the assumption of Theorem 6.2.1, a Doob martingale of $\mathbb{E}_r Y_{\rho^{j_{k-1}}\vee r}^{*L-k+1}$, say $\overline{M}_r^{*L-k+1}$, is determined for $r\geq j_{k-1}$ by the increments*

$$\overline{M}_r^{*L-k+1} - \overline{M}_{j_{k-1}}^{*L-k+1} = M_r^{*L-k+1} - M_{j_{k-1}}^{*L-k+1} + \mathbb{E}_{j_{k-1}} A_{\rho^{j_{k-1}}}^{*L-k+1} - \mathbb{E}_r A_{\rho^{j_{k-1}}}^{*L-k+1}.$$

Proof. By the Doob decomposition we can write

$$
\begin{aligned}
&\mathbb{E}_r Y^{*L-k+1}_{\rho^{j_{k-1}} \vee r} \\
&= \mathbb{1}_{\{r<\rho^{j_{k-1}}\}} \mathbb{E}_r Y^{*L-k+1}_{\rho^{j_{k-1}}} + \mathbb{1}_{\{r \geq \rho^{j_{k-1}}\}} Y^{*L-k+1}_r \\
&= \mathbb{1}_{\{j_{k-1} \leq r<\rho^{j_{k-1}}\}} \left(Y^{*L-k+1}_{j_{k-1}} + M^{*L-k+1}_r - M^{*L-k+1}_{j_{k-1}} - \mathbb{E}_r A^{*L-k+1}_{\rho^{j_{k-1}}} + A^{*L-k+1}_{j_{k-1}} \right) \\
&\quad + \mathbb{1}_{\{r \geq \rho^{j_{k-1}}\}} \left(Y^{*L-k+1}_{j_{k-1}} + M^{*L-k+1}_r - M^{*L-k+1}_{j_{k-1}} - A^{*L-k+1}_r + A^{*L-k+1}_{j_{k-1}} \right) \\
&= Y^{*L-k+1}_{j_{k-1}} + M^{*L-k+1}_r - M^{*L-k+1}_{j_{k-1}} \\
&\quad - \mathbb{1}_{\{j_{k-1} \leq r<\rho^{j_{k-1}}\}} \mathbb{E}_r A^{*L-k+1}_{\rho^{j_{k-1}}} - \mathbb{1}_{\{r \geq \rho^{j_{k-1}}\}} A^{*L-k+1}_r + A^{*L-k+1}_{j_{k-1}} \\
&= Y^{*L-k+1}_{j_{k-1}} + A^{*L-k+1}_{j_{k-1}} - \mathbb{1}_{\{r \geq \rho^{j_{k-1}}\}} \left(A^{*L-k+1}_r - A^{*L-k+1}_{\rho^{j_{k-1}}} \right) \qquad (6.14) \\
&\quad + M^{*L-k+1}_r - M^{*L-k+1}_{j_{k-1}} - \mathbb{E}_r A^{*L-k+1}_{\rho^{j_{k-1}}}.
\end{aligned}
$$

Note that line (6.14) is the sum of a $\mathcal{F}_{k-1}$-measurable random variable and a predictable process. Invoking the Doob martingale representation for $\mathbb{E}_r Y^{*L-k+1}_{\rho^{j_{k-1}} \vee r}$, we obtain

$$
\begin{aligned}
&\sum_{p=j_{k-1}}^{r} \left(M^{*L-k+1}_{p+1} - \mathbb{E}_{p+1} A^{*L-k+1}_{\rho^{j_{k-1}}} - M^{*L-k+1}_p + \mathbb{E}_p A^{*L-k+1}_{\rho^{j_{k-1}}} \right) \\
&\qquad = M^{*L-k+1}_{r+1} - M^{*L-k+1}_{j_{k-1}} - \mathbb{E}_{r+1} A^{*L-k+1}_{\rho^{j_{k-1}}} + \mathbb{E}_{j_{k-1}} A^{*L-k+1}_{\rho^{j_{k-1}}}.
\end{aligned}
$$

This proves the claim. □

We now can prove the dual representation.

Proof of Theorem 6.2.1. ***(i)*** Suppose that for $k = 1, \ldots, L$, M^{L-k+1} is a martingale and A^{L-k+1} is an adapted and integrable process. Then, the process $M_r^{L-k+1,j_1,\ldots,j_{k-1}}$ defined for $r \geq j_{k-1}$ via

$$
M_r^{L-k+1,j_1,\ldots,j_{k-1}} := \prod_{l=1}^{k-1} V^l_{j_l} \left(M^{L-k+1}_r - M^{L-k+1}_{j_{k-1}} + \mathbb{E}_{j_{k-1}} A^{L-k+1}_{\rho^{j_{k-1}}} - \mathbb{E}_r A^{L-k+1}_{\rho^{j_{k-1}}} \right)
$$

is a martingale due to the boundedness of the processes V^l. By Theorem 6.1.1-***(i)***, we have

$$
\begin{aligned}
Y_i^{*L} &\leq \mathbb{E}_i \max_{i \leq j_1 \leq \cdots \leq j_L \leq \partial} \left(X_{j_1,\ldots,j_L} + \sum_{k=1}^{L} \left(M^{L-k+1,j_1,\ldots,j_{k-1}}_{j_{k-1}} - M^{L-k+1,j_1,\ldots,j_{k-1}}_{j_k} \right) \right) \\
&= \mathbb{E}_i \max_{i \leq j_1 \leq \cdots \leq j_L \leq \partial} \Bigg(X_{j_1,\ldots,j_L} \\
&\qquad + \sum_{k=1}^{L} \prod_{l=1}^{k-1} V^l_{j_l} \left(M^{L-k+1}_{j_{k-1}} - M^{L-k+1}_{j_k} + \mathbb{E}_{j_k} A^{L-k+1}_{\rho^{j_{k-1}}} - \mathbb{E}_{j_{k-1}} A^{L-k+1}_{\rho^{j_{k-1}}} \right) \Bigg),
\end{aligned}
$$

with X as defined in (6.7) for sufficiently large $N \in \mathbb{N}$. Letting N tend to infinity, we observe that maximization only takes place over those $j_1 \le \cdots \le j_L$ which satisfy $\mathcal{C}_L(j_1, \ldots, j_L) = 1$. Plugging in the definition of X for those $j_1 \le \cdots \le j_L$ yields the claim.

(ii) We now apply Theorem 6.1.1-***(ii)*** for X as defined in (6.7) with some sufficiently large $N \in \mathbb{N}$. Letting N tend to infinity again and substituting the definition of X, we obtain

$$Y_i^{*L} = \max_{\substack{i \le j_1 \le \cdots \le j_L \le \partial \\ \mathcal{C}_L(j_1,\ldots,j_L)=1}} \left(\sum_{k=1}^{L} U_{j_k}^k \prod_{l=1}^{k-1} V_{j_l}^l + \sum_{k=1}^{L} \left(M_{j_{k-1}}^{*L-k+1,j_1,\ldots,j_{k-1}} - M_{j_k}^{*L-k+1,j_1,\ldots,j_{k-1}} \right) \right),$$

whenever $\left(M_r^{*L-k+1,j_1,\ldots,j_{k-1}}\right)_{r \ge j_{k-1}}$ are the Doob martingales of $\left(Y_r^{*L-k+1,j_1,\ldots,j_{k-1}}\right)_{r \ge j_{k-1}}$. By Proposition 6.2.2-***(i)*** and Proposition 6.2.3 we can take

$$\begin{aligned} & M_{j_{k-1}}^{*L-k+1,j_1,\ldots,j_{k-1}} - M_{j_k}^{*L-k+1,j_1,\ldots,j_{k-1}} \\ &= \prod_{l=1}^{k-1} V_{j_l}^l \left(M_{j_{k-1}}^{*L-k+1} - M_{j_k}^{*L-k+1} + \mathbb{E}_{j_k} A_{\rho^{j_{k-1}}}^{*L-k+1} - \mathbb{E}_{j_{k-1}} A_{\rho^{j_{k-1}}}^{*L-k+1} \right). \end{aligned}$$

This finishes the proof. □

Theorem 6.2.1 gives a straightforward generic way to calculate upper bounds for multiple stopping problems of the form (6.6) at time $i = 0$ via Monte Carlo simulation by performing the following steps in a Markovian setting:

1. Solve the dynamic program from Proposition 6.2.1 for the auxiliary problems Y^{*L-k+1} approximately, and let $\hat{Y}^{L-k+1}$, $k = 1, \ldots, L$, denote the respective approximations.

2. Perform the Doob decomposition of $\hat{Y}^{L-k+1}$, $k = 1, \ldots, L$, numerically, e.g. by one layer of nested Monte Carlo as suggested by Andersen and Broadie [2] in the context of options with a single early exercise right.

3. Plug the processes which stem from the numerical Doob decomposition into the formula from Theorem 6.2.1-***(i)*** and replace the outer expectation by the sample mean.

This agenda will be carried out in more detail in Section 6.3 in the context of swing options. Notice that for a large maturity and a large number of exercise rights, the pathwise maximum in the dual representation of Theorem 6.2.1 runs over a huge set. We now show that due to the special structure of the payoff in (6.6), this maximum can be computed efficiently by a recursion over the time steps and exercise levels.

To this end, given any L-tuple of martingales $M = (M^1, \ldots, M^L)$ and any L-tuple of adapted processes $A = (A^1, \dots, A^L)$, define for $n = 0, \dots, L$ and $i = 0, \dots, \partial$

$$\theta_i^{n,L}(M, A) := \max_{\substack{j_0 = i \le j_1 \le \dots \le j_{L-n} \\ \mathcal{C}_{L-n}(j_1, \dots, j_{L-n}) = 1}} \sum_{k=1}^{L-n} \left(\prod_{l=1}^{k-1} V_{j_l}^{l+n} \right) \Big(U_{j_k}^{n+k} - \Big(M_{j_k}^{L-n-k+1} - M_{j_{k-1}}^{L-n-k+1} \Big) \\ + \mathbb{1}_{\{k>1 \,\wedge\, j_k > j_{k-1}\}} \big(A_{\rho^{j_{k-1}}}^{L-k-n+1} - \mathbb{E}_{j_{k-1}} A_{\rho^{j_{k-1}}}^{L-k-n+1} \big) \Big).$$

By Theorem 6.2.1, we have

$$Y_0^{*L} \le \mathbb{E}[\theta_0^{0,L}(M, A)]$$

for any pair of L-tuples (M, A), and

$$Y_0^{*L} = \theta_0^{0,L}(M^*, A^*)$$

holds for an optimal pair of L-tuples (M^*, A^*). Generalizing a related formula from Balder et al. [8] which deals with the pricing of flexible (or chooser) caps, the expression $\theta_0^{0,L}(M, A)$ can be recursively calculated by the following proposition.

Proposition 6.2.4. *For every L-tuple of martingales $M = (M^1, \dots, M^L)$ and adapted processes $A = (A^1, \dots, A^L)$ and for $i = 0, \dots, T$ and $n = 0, \dots, L$, we have*

$$\theta_i^{n,L}(M, A) = \max \Big\{ \theta_{i+1}^{n,L}(M, A) - (M_{i+1}^{L-n} - M_i^{L-n}), \max_{\nu = 1, \dots, v_i \wedge (L-n)} \sum_{k=1}^{\nu} \left(\prod_{l=1}^{k-1} V_i^{l+n} \right) U_i^{n+k} \\ + \left(\prod_{\lambda=1}^{\nu} V_i^{\lambda+n} \right) \Big(\theta_{\rho^i}^{n+\nu,L}(M, A) - (M_{\rho^i}^{L-n-\nu} - M_i^{L-n-\nu}) \\ + A_{\rho^i}^{L-n-\nu} - \mathbb{E}_i A_{\rho^i}^{L-n-\nu} \Big) \Big\},$$

with

$$\theta_\partial^{n,L}(M, A) = \sum_{k=1}^{L-n} \left(\prod_{l=1}^{\nu} V_\partial^{l+n} \right) U_\partial^{n+k}.$$

Proof. The formula for $\theta_\partial^{n,L}(M, A)$ is obvious by definition. In order to prove the recursive formula, we denote

$$F^{n,L}(j_0, \dots, j_{L-n}) = \sum_{k=1}^{L-n} \left(\prod_{l=1}^{k-1} V_{j_l}^{l+n} \right) \Big(U_{j_k}^{n+k} - (M_{j_k}^{L-n-k+1} - M_{j_{k-1}}^{L-n-k+1}) \\ + \mathbb{1}_{\{k>1 \,\wedge\, j_k > j_{k-1}\}} \big(A_{\rho^{j_{k-1}}}^{L-k-n+1} - \mathbb{E}_{j_{k-1}} A_{\rho^{j_{k-1}}}^{L-k-n+1} \big) \Big).$$

Then, we have

$$\theta_i^{n,L}(M, A) = \max_{\nu = 0, \dots, v_i \wedge (L-n)} \Big\{ \max_{\substack{j_0 = \dots = j_\nu = i < j_{\nu+1} \le \dots \le j_{L-n} \\ \mathcal{C}_{L-n}(j_1, \dots, j_{L-n}) = 1}} F^{n,L}(j_0, \dots, j_{L-n}) \Big\}. \tag{6.15}$$

For $\nu = 0$, we get

$$
\begin{aligned}
&\max_{\substack{j_0=i<j_1\le\cdots\le j_{L-n}\\ \mathcal{C}_{L-n}(j_1,\dots,j_{L-n})=1}} F^{n,L}(j_0,\dots,j_{L-n})\\
&= \max_{\substack{j_0=i+1\le j_1\le\cdots\le j_{L-n}\\ \mathcal{C}_{L-n}(j_1,\dots,j_{L-n})=1}} F^{n,L}(j_0,\dots,j_{L-n}) - (M_{i+1}^{L-n} - M_i^{L-n})\\
&= \theta_{i+1}^{n,L}(M,A) - (M_{i+1}^{L-n} - M_i^{L-n}).
\end{aligned}
\tag{6.16}
$$

For $\nu > 0$, we obtain

$$
\begin{aligned}
&\max_{\substack{j_0=\cdots=j_\nu=i<j_{\nu+1}\le\cdots\le j_{L-n}\\ \mathcal{C}_{L-n}(j_1,\dots,j_{L-n})=1}} F^{n,L}(j_0,\dots,j_{L-n})\\
&= \sum_{k=1}^{\nu}\left(\prod_{l=1}^{k-1} V_i^{l+n}\right) U_i^{n+k}\\
&\quad + \left(\prod_{\lambda=1}^{\nu} V_i^{\lambda+n}\right) \max_{\substack{j_\nu=i,\,\rho^i\le j_{\nu+1}\le\cdots\le j_{L-n}\\ \mathcal{C}_{L-n-\nu}(j_{\nu+1},\dots,j_{L-n})=1}} \sum_{k=\nu+1}^{L-n}\left(\prod_{l=\nu+1}^{k-1} V_{j_l}^{l+n}\right)\Big(U_{j_k}^{n+k} - M_{j_k}^{L-n-k+1}\\
&\qquad + M_{j_{k-1}}^{L-n-k+1} + \mathbb{1}_{\{j_k>j_{k-1}\}}\big(A_{\rho^{j_{k-1}}}^{L-k-n+1} - \mathbb{E}_{j_{k-1}} A_{\rho^{j_{k-1}}}^{L-k-n+1}\big)\Big)\\
&= \sum_{k=1}^{\nu}\left(\prod_{l=1}^{k-1} V_i^{l+n}\right) U_i^{n+k}\\
&\quad + \left(\prod_{\lambda=1}^{\nu} V_i^{\lambda+n}\right)\Big(\big(A_{\rho^i}^{L-\nu-n} - \mathbb{E}_i A_{\rho^i}^{L-\nu-n}\big) - \big(M_{\rho^i}^{L-\nu-n} - M_i^{L-\nu-n}\big)\Big)\\
&\quad + \left(\prod_{\lambda=1}^{\nu} V_i^{\lambda+n}\right) \max_{\substack{j_\nu=\rho^i\le j_{\nu+1}\le\cdots\le j_{L-n}\\ \mathcal{C}_{L-n-\nu}(j_{\nu+1},\dots,j_{L-n})=1}} \sum_{k=\nu+1}^{L-n}\left(\prod_{l=\nu+1}^{k-1} V_{j_l}^{l+n}\right)\Big(U_{j_k}^{n+k} - M_{j_k}^{L-n-k+1}\\
&\qquad + M_{j_{k-1}}^{L-n-k+1} + \mathbb{1}_{\{k>\nu+1\,\wedge\, j_k>j_{k-1}\}}\big(A_{\rho^{j_{k-1}}}^{L-k-n+1} - \mathbb{E}_{j_{k-1}} A_{\rho^{j_{k-1}}}^{L-k-n+1}\big)\Big)\\
&= \sum_{k=1}^{\nu}\left(\prod_{l=1}^{k-1} V_i^{l+n}\right) U_i^{n+k}\\
&+ \left(\prod_{\lambda=1}^{\nu} V_i^{\lambda+n}\right)\Big(\big(A_{\rho^i}^{L-\nu-n} - \mathbb{E}_i A_{\rho^i}^{L-\nu-n}\big) - \big(M_{\rho^i}^{L-\nu-n} - M_i^{L-\nu-n}\big)\Big)\\
&+ \left(\prod_{\lambda=1}^{\nu} V_i^{\lambda+n}\right) \max_{\substack{j_0=\rho^i\le j_1\le\cdots\le j_{L-n-\nu}\\ \mathcal{C}_{L-n-\nu}(j_1,\dots,j_{L-n-\nu})=1}} \sum_{k=1}^{L-n-\nu}\left(\prod_{l=1}^{k-1} V_{j_l}^{l+n+\nu}\right)\Big(U_{j_k}^{n+\nu+k} - M_{j_k}^{L-n-\nu-k+1}\\
&\qquad + M_{j_{k-1}}^{L-n-\nu-k+1} + \mathbb{1}_{\{k>1\,\wedge\, j_k>j_{k-1}\}}\big(A_{\rho^{j_{k-1}}}^{L-k-n-\nu+1} - \mathbb{E}_{j_{k-1}} A_{\rho^{j_{k-1}}}^{L-k-n-\nu+1}\big)\Big).
\end{aligned}
$$

This yields

$$
\begin{aligned}
&\max_{\substack{j_0=\cdots=j_\nu=i<j_{\nu+1}\le\cdots\le j_{L-n}\\ \mathcal{C}_{L-n}(j_1,\ldots,j_{L-n})=1}} F^{n,L}(j_0,\ldots,j_{L-n})\\
&=\sum_{k=1}^{\nu}\left(\prod_{l=1}^{k-1}V_i^{l+n}\right)U_i^{n+k}\\
&\quad+\left(\prod_{\lambda=1}^{\nu}V_i^{\lambda+n}\right)\Big((A_{\rho^i}^{L-\nu-n}-\mathbb{E}_iA_{\rho^i}^{L-\nu-n})-(M_{\rho^i}^{L-\nu-n}-M_i^{L-\nu-n})\Big)\\
&\quad+\left(\prod_{\lambda=1}^{\nu}V_i^{\lambda+n}\right)\theta_{\rho^i}^{n+\nu,L}(M,A).
\end{aligned}
$$

Plugging this identity and (6.16) into (6.15) finishes this technical proof. □

6.2.2 Dual representation based on Snell envelopes

In this subsection we present a simplified version of the dual representation from Corollary 6.1.1 in terms of approximate Snell envelopes for the multiple stopping problem of the form (6.6). It avoids the parameterization in terms of the j_k's.

Theorem 6.2.2. *Suppose Y_i^{*L} is given by (6.6) for some fixed $0\le i\le\partial$. Let $(Y^k)_{1\le k\le L}$ be any set of integrable approximations to $(Y^{*k})_{1\le k\le L}$ defined in* (6.8). *With the conventions $j_0:=-1$, $\rho^{j_0}:=i$, and $Y^0=0$, we then have,*

$$
\begin{aligned}
&Y_i^{*L}-Y_i^L\\
&\le\mathbb{E}_i\max_{\substack{i\le j_1\le\cdots\le j_L\le\partial,\\ \mathcal{C}_L(j_1,\ldots,j_L)=1}}\sum_{k=1}^{L}\left\{\sum_{r=\rho^{j_{k-1}}}^{j_k-1}\prod_{l=1}^{k-1}V_{j_l}^l\left(\mathbb{E}_rY_{r+1}^{L-k+1}-Y_r^{L-k+1}\right)\right.\\
&\quad+\mathbb{1}_{\{j_k>j_{k-1}\}}\prod_{l=1}^{k-1}V_{j_l}^l\Bigg(\max_{1\le n\le v_{j_k}\wedge(L-k+1)}\Bigg\{\sum_{p=k}^{k+n-1}U_{j_k}^p\prod_{l=k}^{p-1}V_{j_k}^l\\
&\quad\left.+\prod_{l=k}^{k+n-1}V_{j_k}^l\mathbb{E}_{j_k}Y_{\rho^{j_k}}^{L-k-n+1}\Bigg\}-Y_{j_k}^{L-k+1}\Bigg)\right\}.
\end{aligned}
$$

Moreover, the righthand side becomes zero if $Y^k=Y^{*k}$, for $k=1,\ldots,L$.

Proof. Suppose $0\le i\le\partial$ is fixed and assume that integrable and adapted processes Y^{L-k+1}, $k=1,\ldots,L$, are given which we consider as approximations of the Snell envelopes of the auxiliary multiple stopping problems Y^{*L-k+1}. Following the relationships for the Snell envelopes Y^{*L-k+1} and $Y^{*L-k+1,j_1,\ldots,j_{k-1}}$ from Proposition 6.2.2, we define for $k>1$ approximations to $Y^{*L-k+1,j_1,\ldots,j_{k-1}}$ via

$$
Y_r^{L-k+1,j_1,\ldots,j_{k-1}}:=\sum_{p=1}^{k-1}U_{j_p}^p\prod_{l=1}^{p-1}V_{j_l}^l+\mathbb{E}_rY_{\rho^{j_{k-1}}\vee r}^{L-k+1}\prod_{l=1}^{k-1}V_{j_l}^l,\quad r>j_{k-1},\tag{6.17}
$$

and for $r = j_{k-1}$

$$\begin{aligned}
&Y_{j_{k-1}}^{L-k+1,j_1,\ldots,j_{k-1}} \\
&:= \sum_{p=1}^{k-1} U_{j_p}^p \prod_{l=1}^{p-1} V_{j_l}^l \\
&\quad + \prod_{l=1}^{k-1} V_{j_l}^l \max_{n\in N(j_1,\ldots,j_{k-1})} \left\{ \sum_{p=k}^{k-1+n} U_{j_{k-1}}^p \prod_{l=k}^{p-1} V_{j_{k-1}}^l + \prod_{l=k}^{k-1+n} V_{j_{k-1}}^l \mathbb{E}_{j_{k-1}} Y_{\rho^{j_{k-1}}}^{L-k+1-n} \right\}. \qquad (6.18)
\end{aligned}$$

Moreover, we define $Y^{L,\varnothing} = Y^L$ for $k = 1$.

Applying Corollary 6.1.1 for X as defined in (6.6) and the above approximations we obtain

$$\begin{aligned}
Y_i^{*L} \le Y_i^L + \mathbb{E}_i \max_{\substack{i\le j_1\le\cdots\le j_L\le\partial,\\ \mathcal{C}_L(j_1,\ldots,j_L)=1}} \sum_{k=1}^{L} \Bigg(& Y_{j_k}^{L-k,j_1,\ldots,j_k} - Y_{j_k}^{L-k+1,j_1,\ldots,j_{k-1}} \\
& + \sum_{l=j_{k-1}}^{j_k-1} \left(\mathbb{E}_l Y_{l+1}^{L-k+1,j_1,\ldots,j_{k-1}} - Y_l^{L-k+1,j_1,\ldots,j_{k-1}} \right) \Bigg), \qquad (6.19)
\end{aligned}$$

where we again observe that the pathwise maximum is attained on the set $\mathcal{C}_L(j_1,...,j_L) = 1$ by letting N (in the definition of X) tend to infinity.

In order to prove the upper bound, it is, in view of (6.19), sufficient to show that, for $i \le j_1 \le \cdots \le j_L \le \partial$ with $\mathcal{C}_L(j_1,...,j_L) = 1$ the following assertions are true:

(i) If $k = 2,\ldots,L$ and $j_k > j_{k-1}$ or if $k = 1$, then

$$\begin{aligned}
&Y_{j_k}^{L-k,j_1,\ldots,j_k} - Y_{j_k}^{L-k+1,j_1,\ldots,j_{k-1}} \\
&= \prod_{l=1}^{k-1} V_{j_l}^l \left(\max_{0\le n\le (v_{j_k}-1)\wedge(L-k)} \left\{ \sum_{p=k}^{k+n} U_{j_k}^p \prod_{l=k}^{p-1} V_{j_k}^l + \prod_{l=k}^{k+n} V_{j_k}^l \mathbb{E}_{j_k} Y_{\rho^{j_k}}^{L-k-n} \right\} - Y_{j_k}^{L-k+1} \right).
\end{aligned}$$

(ii) If $k = 2,\ldots,L$ and $j_k = j_{k-1}$, then

$$Y_{j_k}^{L-k,j_1,\ldots,j_k} - Y_{j_k}^{L-k+1,j_1,\ldots,j_{k-1}} \le 0.$$

(iii) If $k = 1$ and $i \le r \le j_1 - 1$, or if $k = 2,\ldots,L$ and $\rho^{j_{k-1}} \le r \le j_k - 1$, then

$$\mathbb{E}_r Y_{r+1}^{L-k+1,j_1,\ldots,j_{k-1}} - Y_r^{L-k+1,j_1,\ldots,j_{k-1}} = \prod_{l=1}^{k-1} V_{j_l}^l \left(\mathbb{E}_r Y_{r+1}^{L-k+1} - Y_r^{L-k+1} \right).$$

(iv) For $k = 2,\ldots,L$ and $j_{k-1} \le r < \rho^{j_{k-1}}$

$$\mathbb{E}_r Y_{r+1}^{L-k+1,j_1,\ldots,j_{k-1}} - Y_r^{L-k+1,j_1,\ldots,j_{k-1}} \le 0.$$

We first show (i). To this end suppose that $k \ge 2$ and $j_k > j_{k-1}$. Then we have $\mathcal{E}_k(j_1,\ldots,j_k) = 1$ which implies $N(j_1,\ldots,j_k) = \{n;\ 0 \le n \le (v_k - 1) \wedge (L-k)\}$. Hence, by

(6.18),

$$
\begin{aligned}
&Y_{j_k}^{L-k,j_1,\dots,j_k} \\
&= \sum_{p=1}^{k-1} U_{j_p}^p \prod_{l=1}^{p-1} V_{j_l}^l + \prod_{l=1}^{k-1} V_{j_l}^l \max_{0\le n\le (v_{j_k}-1)\wedge(L-k)} \left\{ \sum_{p=k}^{k+n} U_{j_k}^p \prod_{l=k}^{p-1} V_{j_k}^l + \prod_{l=k}^{k+n} V_{j_k}^l \mathbb{E}_{j_k} Y_{\rho^{j_k}}^{L-k-n} \right\}.
\end{aligned}
$$

Subtracting the defining equation (6.17) for $Y_{j_k}^{L-k+1,j_1,\dots,j_{k-1}}$ from the above expression, we obtain (i) because $j_k \ge \rho^{j_{k-1}}$. For $k=1$, we get $\mathcal{E}_1(j_1)=1$ and (i) follows in the same way, taking the definition $Y_{j_1}^{L,\varnothing} = Y_{j_1}^L$ into account.

In order to derive (ii), we note that $\mathcal{E}_k(j_1,\dots,j_k) = \mathcal{E}_{k-1}(j_1,\dots,j_{k-1})+1$ for $j_k = j_{k-1}$. Thus, we have

$$
\begin{aligned}
&Y_{j_k}^{L-k,j_1,\dots,j_k} \\
&= \sum_{p=1}^{k-1} U_{j_p}^p \prod_{l=1}^{p-1} V_{j_l}^l \\
&\quad + \prod_{l=1}^{k-1} V_{j_l}^l \max_{0\le n\le (v_{j_k}-\mathcal{E}_{k-1}(j_1,\dots,j_{k-1})-1)\wedge(L-k)} \left\{ \sum_{p=k}^{k+n} U_{j_k}^p \prod_{l=k}^{p-1} V_{j_k}^l + \prod_{l=k}^{k+n} V_{j_k}^l \mathbb{E}_{j_k} Y_{\rho^{j_k}}^{L-k-n} \right\} \\
&\le \sum_{p=1}^{k-1} U_{j_p}^p \prod_{l=1}^{p-1} V_{j_l}^l \\
&\quad + \prod_{l=1}^{k-1} V_{j_l}^l \max_{0\le n\le (v_{j_{k-1}}-\mathcal{E}_{k-1}(j_1,\dots,j_{k-1}))\wedge(L-k+1)} \left\{ \sum_{p=k}^{k+n} U_{j_{k-1}}^p \prod_{l=k}^{p-1} V_{j_{k-1}}^l + \prod_{l=k}^{k+n} V_{j_{k-1}}^l \mathbb{E}_{j_{k-1}} Y_{\rho^{j_{k-1}}}^{L-k-n} \right\} \\
&= Y_{j_{k-1}}^{L-k+1,j_1,\dots,j_{k-1}} = Y_{j_k}^{L-k+1,j_1,\dots,j_{k-1}}.
\end{aligned}
$$

We next prove (iii). The case $k=1$ is trivial in view of the definition of $Y^{L,\varnothing}$. Hence, we assume that $k\ge 2$ and $\rho^{j_{k-1}} \le r \le j_k - 1$. Then, $r+1 > r \ge \rho^{j_{k-1}} > j_{k-1}$ and by (6.17), we have

$$
\begin{aligned}
\mathbb{E}_r Y_{r+1}^{L-k+1,j_1,\dots,j_{k-1}} &= \sum_{p=1}^{k-1} U_{j_p}^p \prod_{l=1}^{p-1} V_{j_l}^l + \mathbb{E}_r Y_{r+1}^{L-k+1} \prod_{l=1}^{k-1} V_{j_l}^l, \\
Y_r^{L-k+1,j_1,\dots,j_{k-1}} &= \sum_{p=1}^{k-1} U_{j_p}^p \prod_{l=1}^{p-1} V_{j_l}^l + Y_r^{L-k+1} \prod_{l=1}^{k-1} V_{j_l}^l.
\end{aligned}
$$

Taking the difference of both equations yields (iii).

It remains to show (iv). For $k\ge 2$ and $j_{k-1} < r < \rho^{j_{k-1}}$, (6.17) implies

$$
\mathbb{E}_r Y_{r+1}^{L-k+1,j_1,\dots,j_{k-1}} = \sum_{p=1}^{k-1} U_{j_p}^p \prod_{l=1}^{p-1} V_{j_l}^l + \mathbb{E}_r Y_{\rho^{j_{k-1}}}^{L-k+1} \prod_{l=1}^{k-1} V_{j_l}^l = Y_r^{L-k+1,j_1,\dots,j_{k-1}}.
$$

Finally, for $k \geq 2$ and $r = j_{k-1}$, by (6.17) and (6.18),

$$\begin{aligned}
&\mathbb{E}_{j_{k-1}} Y^{L-k+1,j_1,\ldots,j_{k-1}}_{j_{k-1}+1} = \sum_{p=1}^{k-1} U^p_{j_p} \prod_{l=1}^{p-1} V^l_{j_l} + \mathbb{E}_{j_{k-1}} Y^{L-k+1}_{\rho^{j_{k-1}}} \prod_{l=1}^{k-1} V^l_{j_l} \\
&\quad \leq \sum_{p=1}^{k-1} U^p_{j_p} \prod_{l=1}^{p-1} V^l_{j_l} \\
&\qquad + \prod_{l=1}^{k-1} V^l_{j_l} \max_{n \in N(j_1,\ldots,j_{k-1})} \left\{ \sum_{p=k}^{k-1+n} U^p_{j_{k-1}} \prod_{l=k}^{p-1} V^l_{j_{k-1}} + \prod_{l=k}^{k-1+n} V^l_{j_{k-1}} \mathbb{E}_{j_{k-1}} Y^{L-k+1-n}_{\rho^{j_{k-1}}} \right\} \\
&\quad = Y^{L-k+1,j_1,\ldots,j_{k-1}}_{j_{k-1}}.
\end{aligned}$$

Hence, the asserted upper bound for $Y^{*L}_i - Y^L_i$ is shown. This upper bound is zero if $Y^k = Y^{*k}$ for $k = 1, \ldots, L$, because by Proposition 6.2.1, Y^{*L-k+1} is a supermartingale and $Y^{*L-k+1}_{j_k}$ dominates

$$\max_{1 \leq n \leq v_{j_k} \wedge (L-k+1)} \Big\{ \sum_{p=k}^{k+n-1} U^p_{j_k} \prod_{l=k}^{p-1} V^l_{j_k} + \prod_{l=k}^{k+n-1} V^l_{j_k} \mathbb{E}_{j_k} Y^{*L-k-n+1}_{\rho^{j_k}} \Big\}.$$

□

As a spin-off result from Theorem 6.2.2, we may write the following upper bound for $Y^{*,L}_i$ which avoids the computation of the recursive maximum from Proposition 6.2.4 (cf. Schoenmakers [119][Remark 3.3] for a related result in the context of the standard multiple stopping problem).

Corollary 6.2.1. *Suppose that all assumptions and all conventions of Theorem 6.2.2 are in force. Then, we have*

$$\begin{aligned}
&Y^{*L}_i - Y^L_i \\
&\leq \mathbb{E}_i \left\{ \sum_{r=i}^{T-1} \max_{0 \leq k < L} \left(\mathcal{V}^k_{\max} \left(\mathbb{E}_r Y^{L-k}_{r+1} - Y^{L-k}_r \right)^+ \right) + \sum_{k=1}^{L} \mathcal{V}^{k-1}_{\max} \right. \\
&\left. \times \max_{i \leq j \leq \partial} \left(\max_{1 \leq n \leq v_{j_k} \wedge (L-k+1)} \left(\sum_{p=k}^{k+n-1} U^p_j \prod_{l=k}^{p-1} V^l_j \prod_{l=k}^{k+n-1} V^l_j \mathbb{E}_j Y^{L-k-n+1}_{\rho^j} \right) - Y^{L-k+1}_j \right)^+ \right\},
\end{aligned}$$

where

$$\mathcal{V}^k_{\max} := \prod_{l=1}^{k} \max_{j \geq i} V^l_j.$$

Moreover, the right-hand side becomes zero if $Y^k = Y^{*k}$ for $k = 1, \ldots, L$.

Proof. It is straightforward to check that the upper bound in this corollary is actually an upper bound to the right-hand side of the estimate in Theorem 6.2.2. Moreover, the bound is still tight, i.e. that the right-hand side becomes zero if we have $Y^k = Y^{*k}$, for

$k = 1, ..., L$. This follows from the same argument as at the end of the proof of Theorem 6.2.2. □

6.3 A numerical example

We provide a numerical example for the dual representation of multiple stopping problems in the context of swing option pricing. Throughout this section, we assume $i = 0$, i.e. we provide confidence bounds for the swing option price at time 0. More precisely, we consider a stylized swing option, similar to those considered in Meinshausen and Hambly [93] and Bender [11]. In our setting, the holder of a swing option has the right to buy a certain quantity of electricity in the period $j = 0, \dots, T$, for a fixed strike price $K > 0$, subject to the restriction that the option allows up to $L \geq 1$ exercise opportunities under the volume constraints v_j, and where a refraction period has to be taken into account. Here, we choose $T = 50$ and recall that $\partial := T + 1$. The price of electricity, $(S_t)_{t=0,\dots,T}$, is modeled by the following discretized exponential Gaussian Ornstein-Uhlenbeck process

$$\log(S_j) = (1-k)\Big(\log(S_{j-1}) - \mu\Big) + \mu + \sigma\epsilon_j, \;\; S_0 = s_0 > 0, \tag{6.20}$$

where $(\epsilon_j)_{j=1,\dots,T}$ is a family of independent standard normal random variables and the parameters are specified by

$$\sigma = 0.5, \;\; k = 0.9, \;\; \mu = 0, \;\; s_0 = 1.$$

We set $S_\partial = 0$, which means that no penalty is imposed, if the holder of the option does not exercise all rights. The payoff of the swing option is then given by X in (6.6) with

$$\begin{aligned} &V_j^l := 1, \;\; l = 1, \dots, L-1, \;\; j = 0, \dots, \partial, \\ &U_j^p := Z_j := Z(S_j) := (S_j - K)^+, \;\; j = 0, \dots, \partial, \;\; p = 1, \dots, L. \end{aligned}$$

In our numerical study we assume that the strike price is $K = 1$. As volume constraints we consider the situation of a unit volume constraint $v_i = 1$ for $i = 0, \dots, T$ and the situation of an off-peak swing option with $v_i = 1$ on weekdays and $v_i = 2$ on Saturdays and Sundays. The refraction period which we impose is a constant refraction period, i.e. $\rho^i = (i + \delta) \wedge \partial$ for various choices of the constant $\delta \in \mathbb{N}$.

In this Markovian framework, we produce confidence intervals for the price of the swing option at time $i = 0$ by applying the following steps. The procedure below can easily be generalized to the generic cashflow structure, provided that the problem has a Markovian structure. (For notational convenience we only spell out the algorithm for the swing option case.)

6.3.1 Implementation

Step 1: Precompute an approximation of the continuation values. We employ least squares Monte Carlo regression to obtain an approximation to the continuation values

$$C_j^{*1,l}(S_j) := \mathbb{E}\Big[Y_{j+1}^{*l}\big|\mathcal{F}_j\Big] = \mathbb{E}\Big[Y_{j+1}^{*l}\big|S_j\Big],\ C_T^{*1,l}(S_T) = 0,$$
$$C_j^{*\delta,l}(S_j) := \mathbb{E}\Big[Y_{j+\delta}^{*l}\big|\mathcal{F}_j\Big] = \mathbb{E}\Big[Y_{j+\delta}^{*l}\big|S_j\Big],\ C_T^{*\delta,l}(S_T) = 0,$$

with $l = 1,\ldots,L$, where here and in the following $j+1$ and $j+\delta$ are to be understood as $j+1 \wedge \partial$ and $j+\delta \wedge \partial$. Recall that $(Y_j^{*l})_{j=0,\ldots,T}$ is given by the dynamic program from Proposition 6.2.1. We simulate N_1 independent paths $(S_j^m)_{j=0,\ldots,T}^{m=1,\ldots,N_1}$. Choosing as basis functions

$$\psi_1(x) := x, \quad \psi_2(x) := (x-K)^+,$$

we use $(S_j^m)_{j=0,\ldots,T}^{m=1,\ldots,N_1}$ in a straightforward least squares regression procedure to solve the dynamic program approximately, replacing the conditional expectations by the least squares Monte Carlo estimator. This yields approximations to $C_j^{*1,l}(\cdot)$ and $C_j^{*\delta,l}(\cdot)$, denoted by $C_j^{1,l}(\cdot)$ and $C_j^{\delta,l}(\cdot)$.

Step 2: Compute lower bounds. Given the functions $C_j^{1,l}(\cdot)$ and $C_j^{\delta,l}(\cdot)$, we define a (suboptimal) stopping rule $\big(\tau_j^{p,l}\big)_{1\le p\le l}^{1\le l\le L}$ for $0 \le j \le T$ along a given trajectory $(S_j)_{j=0\ldots,T}$ (which we suppress in the notation below) using the following iteration. Here $\tau_j^{p,l}$ is interpreted as the time at which the investor exercises the p-th right, if l rights are left at time j:

$$\begin{aligned}
&\tau_j^{0,l} := j-\delta;\\
&p := k := 0;\\
&\textit{while } (p<l) \quad \textit{do}\\
&\qquad \tau_j^{p+1,l} := \inf\Big\{(\tau_j^{p,l}+\delta)\wedge\partial \le r \le \partial : \max_{1\le n\le v_r\wedge(l-p)}\Big(nZ_r + C_r^{\delta,l-p-n}\Big) \ge C_r^{1,l-p}\Big\},\\
&\qquad s := \tau_j^{p+1,l},\\
&\qquad k := \operatorname{argmax}_{1\le n\le v_s\wedge(l-p)}\Big(nZ_s + C_s^{\delta,l-p-n}\Big),\\
&\qquad \tau_j^{p+1,l} := \tau_j^{p+2,l} := \ldots := \tau_j^{p+k,l} := s,\\
&\qquad p := p+k,\\
&\textit{end}
\end{aligned} \tag{6.21}$$

When $C_j^{1,l}(\cdot)$ and $C_j^{\delta,l}(\cdot)$ are replaced by $C_j^{*1,l}(\cdot)$ and $C_j^{*\delta,l}(\cdot)$, then this family of stopping times is optimal. Hence, $\big(\tau_j^{p,l}\big)_{1\le p\le l}^{1\le l\le L}$ is a good family of stopping times, if the approximations of the continuation values in Step 1 are reasonably close to the true continuation values.

Remark 6.3.1. *(i) In the situation of unit volume constraint (i.e. $v \equiv 1$), the stopping rule* (6.21) *simplifies to* $\tau_j^{0,l} = j - \delta$ *and*

$$\tau_j^{p,l} = \inf\left\{(\tau_j^{p-1,l} + \delta) \wedge \partial \le r \le \partial : Z_r + C_r^{\delta,l-p} \ge C_r^{1,l-p+1}\right\}, \quad 1 \le l \le L, \quad 1 \le p \le l,$$

compare with eqn. (3.7) in Bender [11].
(ii) In the situation of a trivial refraction period (i.e. $\delta = 1$), the above construction of approximate stopping rules is also used in Aleksandrov and Hambly [1].

Setting

$$\underline{Y}_0^l := \mathbb{E}_0 \sum_{p=1}^{l} Z_{\tau_0^{p,l}},\ \underline{Y}_1^l := \mathbb{E}_1 \sum_{p=1}^{l} Z_{\tau_1^{p,l}},\ \underline{Y}_\delta^l := \mathbb{E}_\delta \sum_{p=1}^{l} Z_{\tau_\delta^{p,l}},$$

we have that $\underline{Y}_0^l$ is a lower bound for Y_0^{*l}. By the tower property of the conditional expectation, we also have

$$\mathbb{E}_0 \underline{Y}_1^l = \mathbb{E}_0 \sum_{p=1}^{l} Z_{\tau_1^{p,l}}, \quad \mathbb{E}_0 \underline{Y}_\delta^l = \mathbb{E}_0 \sum_{p=1}^{l} Z_{\tau_\delta^{p,l}}.$$

As for simulations, we generate a new set of N_2 independent paths of the underlying price process, which we again denote, abusing of notation, by $(S_j^m)_{j=0,\ldots,T}^{m=1,\ldots,N_2}$. Along theses N_2 trajectories we compute $\tau_0^{p,l}$ and apply the notation

$$\tau_0^{p,l,m}, \quad 1 \le p \le l,\ 1 \le m \le N_2.$$

Now the lower biased estimate $\widehat{\underline{Y}}_0^l$ for Y_0^{*l} is calculated by averaging over the N_2 realizations of $\sum_{p=1}^{l} Z_{\tau_0^{p,l}}$, i.e.

$$\widehat{\underline{Y}}_0^l = \frac{1}{N_2} \sum_{m=1}^{N_2} \sum_{p=1}^{l} Z(S^m_{\tau_0^{p,l,m}}), \quad 1 \le l \le L. \tag{6.22}$$

Similarly, we also construct approximations

$$\hat{\mathbb{E}}_0 \underline{Y}_1^l = \frac{1}{N_2} \sum_{m=1}^{N_2} \sum_{p=1}^{l} Z(S_{\tau_1^{p,l,m}}), \quad \hat{\mathbb{E}}_0 \underline{Y}_\delta^l = \frac{1}{N_2} \sum_{m=1}^{N_2} \sum_{p=1}^{l} Z(S_{\tau_\delta^{p,l,m}})$$

of $\mathbb{E}_0 \underline{Y}_1^l$ and $\mathbb{E}_0 \underline{Y}_\delta^l$, which we store for later use. For constructing confidence intervals, we also save the empirical standard deviation $\mathsf{stddev}(\widehat{\underline{Y}}_0^l)$.

Step 3: Compute approximations to the Snell envelopes. Using the stopping rule

(6.21), we consider a family of random variables

$$\underline{Y}_j^l := \mathbb{E}_j \sum_{p=1}^{l} Z_{\tau_j^{p,l}}, \ 1 \le l \le L, \ 0 \le j \le T, \tag{6.23}$$

which is an approximation to the Snell envelope $\left(Y_j^{*l}\right)_{0\le j\le T}^{1\le l\le L}$. We apply the following procedure to simulate $\underline{Y}_j^l$:

We simulate a new set of N_3 paths of the underlying $(S_j^m)_{0\le j\le T}^{1\le m\le N_3}$ (abusing the notation again). We refer to these paths as the outer paths. We now fix a pair (m, j) and compute approximations of $\underline{Y}_j^l$, $\mathbb{E}_j \underline{Y}_{j+1}^l$, and $\mathbb{E}_j \underline{Y}_{j+\delta}^l$ along the m-th outer path which are denoted by $\widehat{\underline{Y}}_j^{l,m}$, $\widehat{\mathbb{E}}_j^m \underline{Y}_{j+1}^l$, and $\widehat{\mathbb{E}}_j^m \underline{Y}_{j+\delta}^l$, respectively. In these approximations the conditional expectations are replaced by the sample mean over a set of inner simulations. Hence, for the fixed path S^m and the fixed time point j, we generate N_4 independent sample paths of $(S_r)_{r=j,\dots,T}$ under the conditional law given that $S_j = S_j^m$. These inner paths are denoted by $(\bar{S}_r^\nu)_{r=j,\dots,T}^{\nu=1,\dots,N_4}$, suppressing here and in the following the dependence on (m, j). Along the inner paths $\bar{S}^\nu$ we compute the stopping times $\tau_i^{p,l}$ for $i = j, j+1, j+\delta$ in (6.21) and apply the notation

$$\tau_i^{p,l,\nu}, \ 1 \le p \le l, \ 1 \le l \le L, \ \nu = 1, \dots, N_4.$$

We now define

$$\widehat{\underline{Y}}_j^{l,m} := \widehat{\mathbb{E}}_j^m \sum_{p=1}^{l} Z_{\tau_j^{p,l}} := \frac{1}{N_4} \sum_{\nu=1}^{N_4} \sum_{p=1}^{l} Z\big(\bar{S}^\nu_{\tau_j^{p,l,\nu}}\big).$$

Similarly, we approximate $\mathbb{E}_j \underline{Y}_{j+1}^l$ for the fixed j along the fixed m-th outer path by

$$\widehat{\mathbb{E}}_j^m \underline{Y}_{j+1}^l := \frac{1}{N_4} \sum_{\nu=1}^{N_4} \sum_{p=1}^{l} Z\big(\bar{S}^\nu_{\tau_{j+1}^{p,l,\nu}}\big),$$

taking the tower property of the conditional expectation into account. The approximation $\widehat{\mathbb{E}}_j^m \underline{Y}_{j+\delta}^l$ is obtained analogously.

Remark 6.3.2. *Note that, for $j = 0$ approximations $\widehat{\underline{Y}}_0^l$, $\widehat{\mathbb{E}}_0 \underline{Y}_1^l$, and $\widehat{\mathbb{E}}_0 \underline{Y}_\delta^l$ of $\underline{Y}_0^l$, $\mathbb{E}_0 \underline{Y}_1^l$, and $\mathbb{E}_0 \underline{Y}_\delta^l$ were already obtained based on the N_2-samples in Step 2. As typically $N_2 > N_4$ these approximations are more accurate. Hence, one can perform Step 3 for $j \ge 1$ only and set*

$$\widehat{\underline{Y}}_0^{l,m} := \widehat{\underline{Y}}_0^l, \quad \widehat{\mathbb{E}}_0^m \underline{Y}_1^l := \widehat{\mathbb{E}}_0 \underline{Y}_1^l, \quad \widehat{\mathbb{E}}_0^m \underline{Y}_\delta^l := \widehat{\mathbb{E}}_0 \underline{Y}_\delta^l, \quad m = 1, \dots, M.$$

This trick of applying the more accurate non-nested Monte Carlo simulation of Step 2 at time 0 leads to a significant decrease of the variance in the simulation of the upper bound. This is in the same spirit as the computation of low variance upper bounds for the standard stopping problem from Andersen and Broadie [2].

Step 4: Compute the upper bounds. The Doob decomposition of $\underline{Y}_j^l$ yields the pair $(\underline{M}_j^l, \underline{A}_j^l)$. Note that due to

$$\underline{M}_{i+1}^l - \underline{M}_i^l = \underline{Y}_{i+1}^l - \mathbb{E}_i \underline{Y}_{i+1}^l,$$

and

$$-(\underline{M}_{i+\delta}^l - \underline{M}_i^l) + \underline{A}_{i+\delta}^l - \mathbb{E}_i \underline{A}_{i+\delta}^l = \mathbb{E}_i \underline{Y}_{i+\delta}^l - \underline{Y}_{i+\delta}^l,$$

we can rewrite the recursion formula in Proposition 6.2.4 as

$$\theta_i^{n,L} = \max\left\{\theta_{i+1}^{n,L} + \mathbb{E}_i \underline{Y}_{i+1}^{L-n} - \underline{Y}_{i+1}^{L-n}, \max_{1\le\nu\le v_i\wedge(L-n)} \left(\nu Z_i + \theta_{i+\delta}^{n+\nu,L} + \mathbb{E}_i \underline{Y}_{i+\delta}^{L-n-\nu} - \underline{Y}_{i+\delta}^{L-n-\nu}\right)\right\}.$$

We now introduce approximations $\theta_i^{n,L,m}$ of $\theta_i^{n,L}$ along the m-th outer path of Step 3 by replacing $\underline{Y}_j^l$, $\mathbb{E}_j \underline{Y}_{j+1}^l$, and $\mathbb{E}_j \underline{Y}_{j+\delta}^l$ with their simulated counterparts $\widehat{\underline{Y}}_j^{l,m}$, $\widehat{\mathbb{E}}_j^m \underline{Y}_{j+1}^l$, and $\widehat{\mathbb{E}}_j^m \underline{Y}_{j+\delta}^l$ constructed in Step 3. As simulation based estimate for the upper bound, we use

$$Y_0^{up,L} := \frac{1}{N_3} \sum_{m=1}^{N_3} \theta_0^{0,L,m}.$$

Replacing the conditional expectations by the sample mean in $\theta_0^{0,L,m}$ introduces an additional bias up thanks to Jensen's inequality and the convexity of the maximum. Hence, the estimator $Y_0^{up,L}$ is biased up by Theorem 6.2.1 and Proposition 6.2.4. Finally, a 95% confidence interval on the price of the swing option is given by

$$\left[\widehat{\underline{Y}}_0^L - 1.96 \times \mathsf{stddev}(\widehat{\underline{Y}}_0^L), Y_0^{up,L} + 1.96 \times \mathsf{stddev}(Y_0^{up,L})\right].$$

6.3.2 Numerical results: swing options with unit volume constraints

We now present some numerical results which the above algorithm produces for the swing option contract as specified at the beginning of this section. Let us first consider the situation of a unit volume constraint, i.e. $v_j := 1$ for $j = 0, \dots, T$. We recall that $\delta \in \mathbb{N}$ denotes a constant refraction period. In this setting the dual representation of Theorem 6.2.1 reduces to the one derived in Bender [11]. In this paper the same swing option example is treated numerically but only for up to three exercise rights only. Thanks to the new recursion formula in Proposition 6.2.4 we can now efficiently treat the case of a large number of exercise rights (here up to $L = 10$). Moreover, the upper bound algorithm in Bender [11] differs slightly from the one we propose here. In Bender [11] the upper bound is calculated based on the numerical Doob decomposition of

$$Y_j^l = \max\{Z_j + C_j^{\delta,l-1}, C_j^{1,l}\}$$

δ	L	$\widehat{\underline{Y}}_0^L$	$Y_0^{up,L}$	95% conf. interval	L	$\widehat{\underline{Y}}_0^L$	$Y_0^{up,L}$	95% conf. interval
1	2	3.3116	3.3211	[3.307, 3.322]	3	4.53627	4.54806	[4.531, 4.549]
2	2	3.27513	3.28469	[3.270, 3.285]	3	4.43753	4.45154	[4.432, 4.452]
3	2	3.2525	3.26286	[3.24, 3.264]	3	4.36706	4.38245	[4.362, 4.383]
4	2	3.2313	3.24083	[3.227, 3.242]	3	4.29996	4.31656	[4.295, 4.318]
5	2	3.20906	3.22061	[3.204, 3.221]	3	4.29996	4.31656	[4.295, 4.318]
6	2	3.18613	3.19809	[3.181, 3.199]	3	4.15557	4.17514	[4.150, 4.176]
8	2	3.13625	3.14984	[3.132, 3.151]	3	3.99773	4.01954	[3.992, 4.021]
10	2	3.09022	3.10332	[3.086, 3.104]	3	3.83377	3.8528	[3.828, 3.854]
12	2	3.03874	3.05196	[3.034, 3.053]	3	3.65492	3.67658	[3.650, 3.678]
14	2	2.98727	3.00048	[2.983, 3.001]	3	3.47017	3.49061	[3.465, 3.492]
16	2	2.92751	2.94214	[2.923, 2.943]	3	3.27524	3.29482	[3.270, 3.296]
18	2	2.87368	2.8888	[2.869, 2.890]	3	3.09209	3.11002	[3.087, 3.111]
20	2	2.81521	2.83005	[2.811, 2.831]	3	2.91951	2.93649	[2.915, 2.938]

Table 6.1: *Unit volume constraints ($v_j \equiv 1$).* Numerical results based on the approximation $\widehat{\underline{Y}}_j^l$ to the Snell envelope via the stopping rule (6.21) for two and three exercise rights.

while we here utilize the numerical Doob decomposition of $\underline{Y}_j^l := \mathbb{E}_j \sum_{p=1}^{l} Z_{\tau_j^{p,l}}$.

The choice of simulation parameters in our study is as follows: in Step 1, we choose $N_1 = 1000$ paths for the least squares Monte Carlo regression to approximate the continuation function. In Step 2, the lower bound is simulated using $N_2 = 300000$ paths and in Step 3, we employ $N_3 = 2000$ outer and $N_4 = 100$ inner paths for the computation of the upper bound. Moreover, we use the variance reduction method from Remark 6.3.2.

Table 6.1 depicts the numerical results for the case of two and three exercise rights for a refraction period ranging from 1 to 20. We observe that the relative length of the 95%-confidence intervals is less than 1% in all cases. A comparison with the numerical results in Bender [11] shows that the differences in the upper price estimator based on Y^l and $\underline{Y}^l$ are negligible, but the variance reduction method of Remark 6.3.2 shrinks the confidence interval significantly.

The numerical results for the case of a larger number of exercise rights ($L = 4, 6, 8, 10$) are presented in Table 6.2. Due to the time horizon of 50 days, it may happen that, for a large number of rights and a large refraction period, some exercise rights cannot be used by the investor. This explains why e.g. the price bounds for the swing option with refraction period $\delta = 14$ are the same for $L = 4$ and $L = 6$ rights. Concerning the accuracy of our numerical procedure we emphasize that the relative difference between lower and upper bound is still less than 1% even in the case of 10 exercise rights.

δ	L	$\widehat{\underline{Y}}_0^L$	$Y_0^{up,L}$	95% conf. interval	L	$\widehat{\underline{Y}}_0^L$	$Y_0^{up,L}$	95% conf. interval
1	4	5.60136	5.614	[5.595, 5.615]	6	7.38677	7.40107	[7.379, 7.402]
2	4	5.41347	5.43091	[5.407, 5.432]	6	6.94554	6.97364	[6.938, 6.975]
3	4	5.27248	5.29342	[5.266, 5.295]	6	6.58739	6.62054	[6.580, 6.622]
4	4	5.13119	5.15543	[5.125, 5.157]	6	6.2151	6.25165	[6.208, 6.254]
5	4	4.98353	5.01117	[4.978, 5.013]	6	5.81445	5.85591	[5.808, 5.858]
6	4	4.82479	4.85239	[4.819, 4.854]	6	5.4079	5.44248	[5.401, 5.444]
8	4	4.49057	4.51822	[4.485, 4.520]	6	4.68469	4.71574	[4.679, 4.718]
10	4	4.13658	4.16231	[4.131, 4.164]	6	4.1662	4.19164	[4.160, 4.193]
12	4	3.78981	3.81429	[3.784, 3.816]	6	3.78992	3.81448	[3.785, 3.816]
14	4	3.50023	3.52138	[3.495, 3.523]	6	3.50023	3.52138	[3.495, 3.523]
1	8	8.83286	8.84907	[8.824, 8.850]	10	10.0219	10.0391	[10.01, 10.04]
2	8	8.04508	8.08421	[8.037, 8.086]	10	8.80264	8.85093	[8.794, 8.853]
3	8	7.36943	7.41474	[7.362, 7.417]	10	7.73096	7.78117	[7.723, 7.783]
4	8	6.66726	6.71129	[6.660, 6.713]	10	6.74649	6.78596	[6.739, 6.788]
5	8	5.99864	6.0388	[5.992, 6.041]	10	6.0035	6.04305	[5.996, 6.045]
6	8	5.45188	5.48518	[5.445, 5.487]	10	5.45187	5.48518	[5.445, 5.487]

Table 6.2: *Unit volume constraints ($v_j \equiv 1$).* Numerical results based on the approximation $\widehat{\underline{Y}}_j^l$ to the Snell envelope via the stopping rule (6.21) for a higher number of exercise rights.

6.3.3 Numerical results: off-peak swing option

We now consider a swing option which allows for buying at most one package of electricity on weekdays and two packages on Saturdays and Sundays (off-peak period). Hence, we have for $j = 0, \ldots, 50$ the volume constraints

$$v_j := \begin{cases} 1, & \text{if } j \text{ is a week day,} \\ 2, & \text{if } j \text{ is a weekend day,} \end{cases} \tag{6.24}$$

where we start in $j = 0$ on a Monday.

We run the above algorithm with $N_1 = 10000$, $N_2 = 300000$, $N_3 = 2000$, and $N_4 = 100$ sample paths. The numerical results for this off-peak swing option are presented in Table 6.3 for various choices of the number L of exercise rights and the length δ of the refraction period. Notice that the dual representations yet available in the literature do not cover the case of a non-trivial refraction period ($\delta \neq 1$) in combination with non-trivial volume constraints ($v \neq 1$). Due to the feature of allowing for exercising twice on weekends, the swing option prices are now higher than in the example with unit volume constraint. Moreover, additional rights can now become beneficial in situations in which they could not be exercised under the unit volume constraint (e.g. the additional 8-th right when the refraction period is $\delta = 14$). As for accuracy, we again observe that the relative length of the 95% confidence interval is less than 1% in all cases, which demonstrates that the algorithm performs equally well in the presence of volume constraints.

In the case of unit refraction period $\delta = 1$, upper price bounds for the off-peak swing option can also be computed by the dual representation of Bender [12] for the marginal price of a multiple exercise option. This approach generalizes the ideas of Meinshausen and Hambly [93]: An upper biased estimate for the marginal price of having an additional lth right is computed in terms of one martingale and $(l-1)$ stopping times. By summing up these upper bounds for the marginal prices, one finally ends with an upper biased estimate for the option price. This approach is based on the fact that, roughly speaking, under the assumption of a trivial refraction period ($\delta = 1$) optimal exercise times for the problem with $(l-1)$ rights are also optimal for the problem with l rights, if one adds one additional exercise time in a clever way. This is clearly not possible in general in the presence of a non-trivial refraction period. So it seems that this alternative approach cannot be easily generalized to include refraction periods.

Table 6.4 compares the upper bounds obtained using our method and the method from Bender [12] for the unit refraction case $\delta = 1$. We mention that in Table 6.4, the variance reduction method from Remark 6.3.2 is not applied for both algorithms. As both methods are run with the same number of sample paths and the nested maximum in our method can be efficiently calculated by the recursion formula from Proposition 6.2.4, the computational effort is roughly the same for both algorithms. We observe that, as the number of exercise rights increases, our method of directly tackling the Snell envelope produces upper bounds that become lower than the algorithm tackling the marginal values from Bender [12]. Whereas the differences for $L = 1, \ldots, 4$ are numerically not significant yet, they however become noticeable starting from $L = 5$ and are striking for

δ	L	$\underline{\widehat{Y}}_0^L$	$Y_0^{up,L}$	95% confidence interval	L	$\underline{\widehat{Y}}_0^L$	$Y_0^{up,L}$	95% confidence interval
1	2	3.39804	3.40779	[3.39342, 3.409]	3	4.722	4.735	[4.716, 4.736]
2	2	3.36368	3.37544	[3.35908, 3.37676]	3	4.635	4.653	[4.629, 4.654]
3	2	3.34728	3.35873	[3.34265, 3.36017]	3	4.581	4.597	[4.575, 4.599]
4	2	3.32763	3.34072	[3.32302, 3.34223]	3	4.523	4.544	[4.517, 4.546]
5	2	3.30829	3.32101	[3.30366, 3.32246]	3	4.464	4.486	[4.458, 4.488]
6	2	3.28626	3.29915	[3.28162, 3.30067]	3	4.402	4.425	[4.397, 4.427]
8	2	3.24383	3.26104	[3.23921, 3.26284]	3	4.292	4.317	[4.286, 4.319]
10	2	3.20872	3.22244	[3.20409, 3.22404]	3	4.184	4.209	[4.178, 4.211]
12	2	3.16994	3.18626	[3.16529, 3.18807]	3	4.066	4.093	[4.060, 4.096]
14	2	3.12722	3.14467	[3.12253, 3.14666]	3	3.953	3.976	[3.947, 3.978]
16	2	3.08634	3.10272	[3.08169, 3.10464]	3	3.864	3.885	[3.858, 3.888]
18	2	3.05186	3.06793	[3.04713, 3.06991]	3	3.756	3.778	[3.750, 3.780]
20	2	3.01557	3.03044	[3.01088, 3.03233]	3	3.651	3.672	[3.645, 3.674]
1	4	5.89744	5.91312	[5.89078, 5.91464]	6	7.913	7.933	[7.905, 7.935]
2	4	5.73736	5.76003	[5.73078, 5.76192]	6	7.553	7.585	[7.545, 7.587]
3	4	5.62688	5.65097	[5.62027, 5.65305]	6	7.284	7.323	[7.276, 7.326]
4	4	5.51763	5.5468	[5.51105, 5.54915]	6	7.019	7.062	[7.011, 7.065]
5	4	5.40154	5.43295	[5.39496, 5.43546]	6	6.741	6.787	[6.733,6.790]
6	4	5.27976	5.30967	[5.27316, 5.31227]	6	6.451	6.497	[6.444, 6.501]
8	4	5.06733	5.10055	[5.06078, 5.10335]	6	5.940	5.985	[5.932, 5.989]
10	4	4.85039	4.88637	[4.84386, 4.88953]	6	5.466	5.507	[5.459, 5.511]
12	4	4.63227	4.66583	[4.62575, 4.66884]	6	5.084	5.117	[5.077, 5.120]
14	4	4.43104	4.45997	[4.42453, 4.46279]	6	4.767	4.799	[4.760, 4.802]
16	4	4.25079	4.27955	[4.24441, 4.28237]	6	4.395	4.423	[4.388, 4.425]
18	4	4.07804	4.10338	[4.07164, 4.10605]	6	4.181	4.207	[4.174, 4.210]
20	4	3.94562	3.96789	[3.93923, 3.97034]	6	4.024	4.047	[4.018, 4.049]
1	8	9.60253	9.62348	[9.59318, 9.62507]	10	11.04	11.06	[11.03, 11.06]
2	8	8.97188	9.01806	[8.96279, 9.02078]	10	10.08	10.14	[10.07, 10.14]
3	8	8.48335	8.53629	[8.47425, 8.53952]	10	9.313	9.37	[9.303, 9.382]
4	8	8.00789	8.06203	[7.99887, 8.06551]	10	8.580	8.638	[8.571, 8.641]
5	8	7.5251	7.57926	[7.5161, 7.58278]	10	7.90	7.96	[7.896, 7.964]
6	8	7.06562	7.11754	[7.05669, 7.12102]	10	7.335	7.384	[7.325, 7.388]
8	8	6.18418	6.23051	[6.17596, 6.23403]	10	6.202	6.247	[6.194, 6.251]
10	8	5.54885	5.58774	[5.54107, 5.59089]	10	5.548	5.587	[5.541, 5.590]

Table 6.3: *Off-peak volume constraints.* Numerical results for the off-peak swing option for various exercise rights and refraction periods. The simulations are based on the approximation $\underline{\widehat{Y}}_j^l$ to the Snell envelope via the stopping rule (6.21).

δ	L	$Y_0^{up,L}$	upper bound using Bender [12]
1	1	1.86485 (0.0019)	1.8638 (0.0019)
1	2	3.40832 (0.003)	3.4078 (0.003)
1	3	4.73509 (0.0037)	4.7368 (0.0038)
1	4	5.90956 (0.0043)	5.9170 (0.0045)
1	5	6.96665 (0.00047)	6.98 (0.0052)
1	6	7.92669 (0.005)	7.9470 (0.0058)
1	7	8.80743 (0.0055)	8.8327 (0.0062)
1	8	9.61643 (0.0058)	9.6493 (0.0069)
1	9	10.3642 (0.0061)	10.4040 (0.0074)
1	10	11.0553 (0.00064)	11.1035 (0.0079)

Table 6.4: *Off-peak volume constraints.* A comparison between our upper bounds and the upper bounds obtained via the algorithm from Bender [12] for the case of unit refraction period. Standard deviations are displayed in parentheses.

e.g. $L = 10$. We also note that the larger L, the better our method performs concerning the variance of the upper bounds. At large, we conclude that if one is mainly interested in the price (and not the marginal price) of the swing option, our new method performs better than the algorithm from Bender [12]. Moreover, it is applicable to a larger class of problems.

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Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Berlin, den 12.12.2012 Jianing Zhang